Fourier Transforms

(Superposition take 3)

EE 206 Circuits I

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Please visit Bison Academy for corresponding lecture notes, homework sets, and solutions

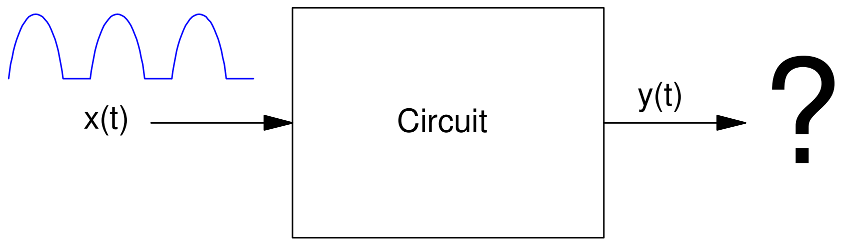
Superposition (review)

Superposition allows you to analyze circuits with multiple sinusoidal inputs. If this is the case

- Treat the problem as N separate problems, each with a single sinusoidal input.
- Solve each of the N problems separately using phasor analysis
- Add up all of the answers to get the total output.

Suppose your circuit has an input that *isn't* a sum of sinusoids.

- One solution is to approximate the input with two sine wave (what we did last lecture)
- A second solution is to define the input in terms of sine waves (this lecture)



Fourier Transform

If x(t) is periodic in time T

$$x(t) = x(t+T)$$

then you can express x(t) as

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

where

$$\omega_0 = \frac{2\pi}{T}$$

Translation:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

Going right to left

- *If you add up a bunch of signals which are periodic in time T, the result is also periodic in time T*
- *Duh.*

Going right to left:

- *If a signal is periodic and is not a sine wave, it is made up of sine waves which are harmonics of the fundamental.*

Finding Fourier Coefficients:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

Analytic Solution: Integration. Wait for ECE 343 to do this

Numeric Solution: Use Matlab

$$a_0 = mean(x)$$

$$a_n = 2 \cdot mean(x \cdot \cos(n\omega_0 t))$$

$$b_n = 2 \cdot mean(x \cdot \sin(n\omega_0 t))$$

Proof: a0:

All sine waves are orthogonal. The DC term is

$$a_0 = mean(x)$$

$$a_0 = mean(a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))$$

$$a_0 = mean(a_0) + mean(a_1 \cos(\omega_0 t)) + mean(a_2 \cos(2\omega_0 t)) + ...$$

The mean of a sine wave is zero

$$a_0 = a_0 + 0 + 0 + ...$$

Proof: a1

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

$$a_1 = 2\ mean(x \cdot \cos(\omega_0 t))$$

$$a_1 = 2\ mean((a_0 + a_1 \cos(\omega_0 t) + b_1 \sin(\omega_0 t) + \ldots) \cdot \cos(\omega_0 t))$$

$$\begin{aligned} a_1 = 2 \cdot mean(a_0 \cdot \cos(\omega_0 t)) \\ +2 \cdot mean(a_1 \cos(\omega_0 t) \cdot \cos(\omega_0 t)) \\ +2 \cdot mean(a_2 \cos(2\omega_0 t) \cdot \cos(\omega_1 t)) \\ +\ldots \end{aligned}$$

The mean of a sine wave is zero. The mean of $\cos^2(t)$ is 1/2

$$a_1 = 0 + 2 \cdot \tfrac{a_1}{2} + 0 + \ldots$$

$$a_1 = a_1$$

etc.

Time Scaling:

Through a change of variable, you can make the period anything you want.

Making the period 2π makes the problem easier:

$$T = 2\pi$$

$$\omega_0 = \frac{2\pi}{T} = 1$$

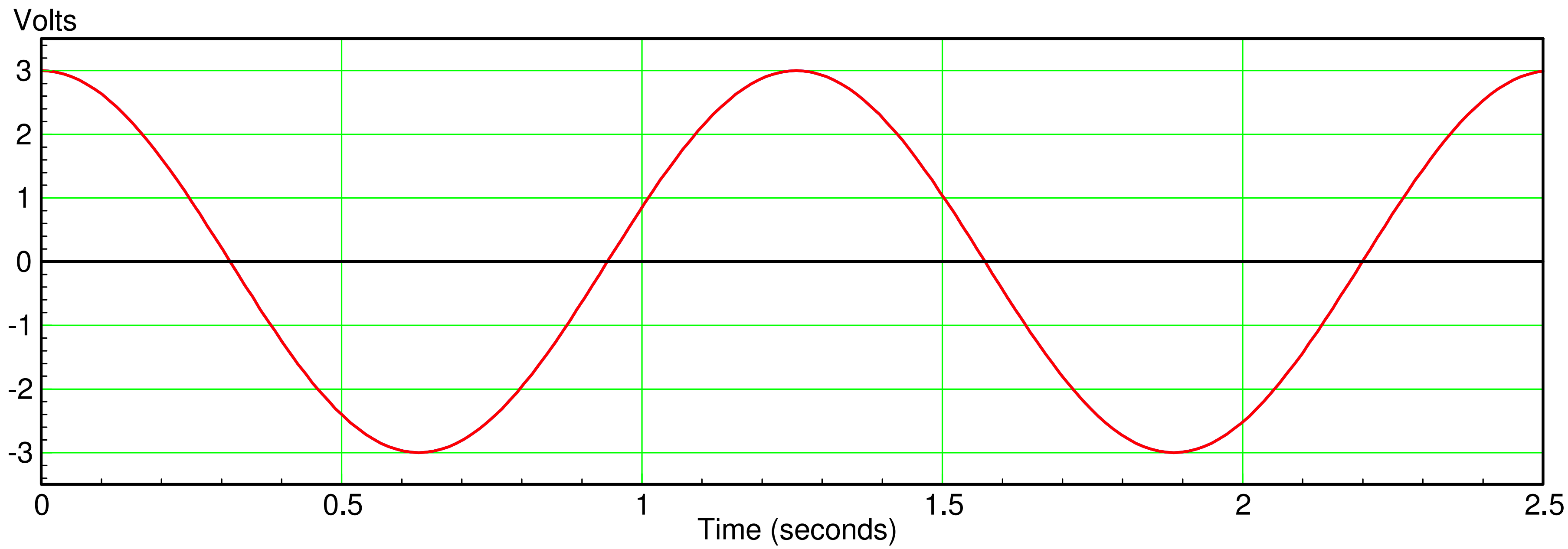
$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt)$$

Common Fourier Transforms: Sine Wave

$$x(t) = 3\cos(5t)$$

Fourier Transform:

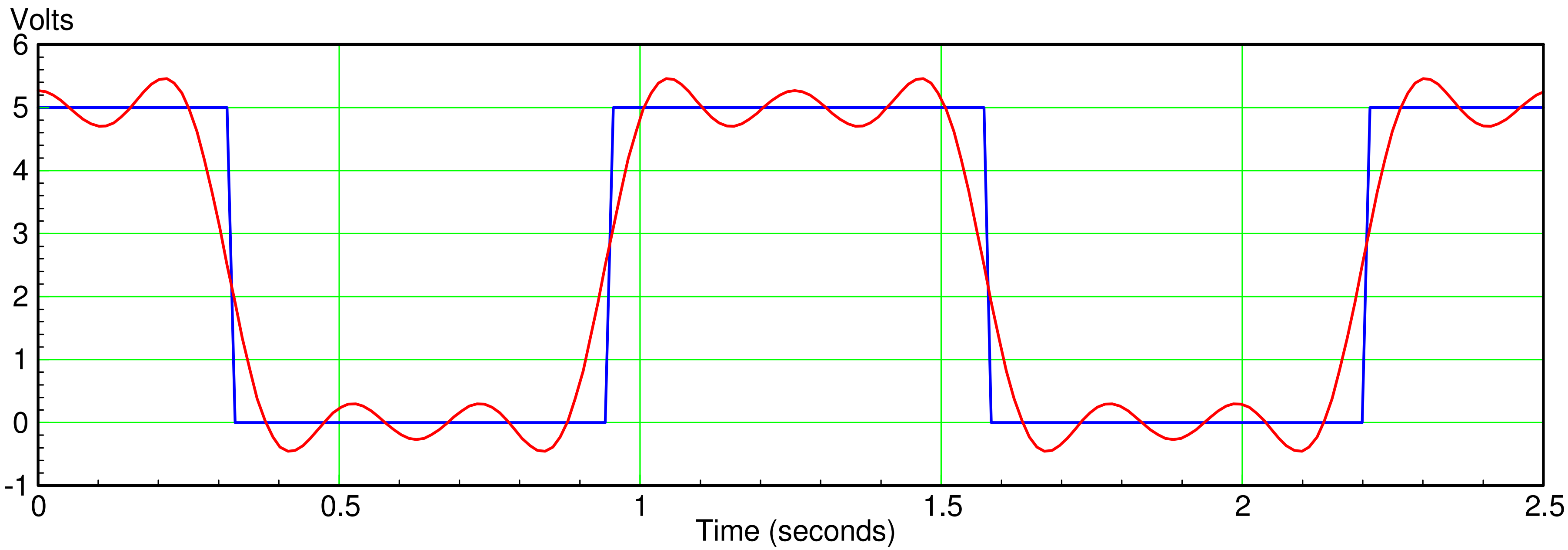
$$x(t) = 3\cos(5t)$$



Square Wave

$$x(t) = \begin{cases} 5V & \cos(5t) > 0 \\ 0V & otherwise \end{cases}$$

$$x(t) \approx 2.5 + 3.18\cos(5t) - 1.05\cos(15t) + 0.64\cos(25t) + \ldots$$

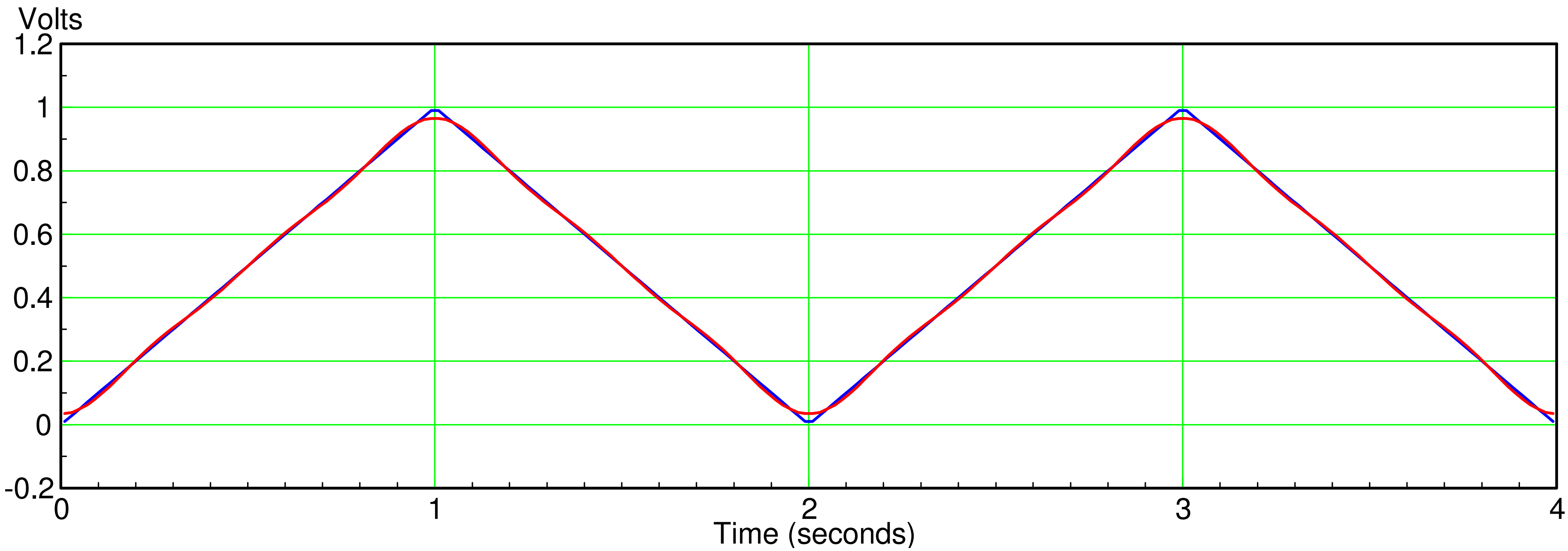


Triangle Wave

$$x(t) = x(t+2)$$

$$x(t) = \begin{cases} t & 0 < t < 1 \\ 2 - t & 1 < t < 2 \end{cases}$$

$$x(t) = 0.5 - 0.405\cos(\pi t) - 0.045\cos(3\pi t) - 0.016\cos(5\pi t) + \ldots$$

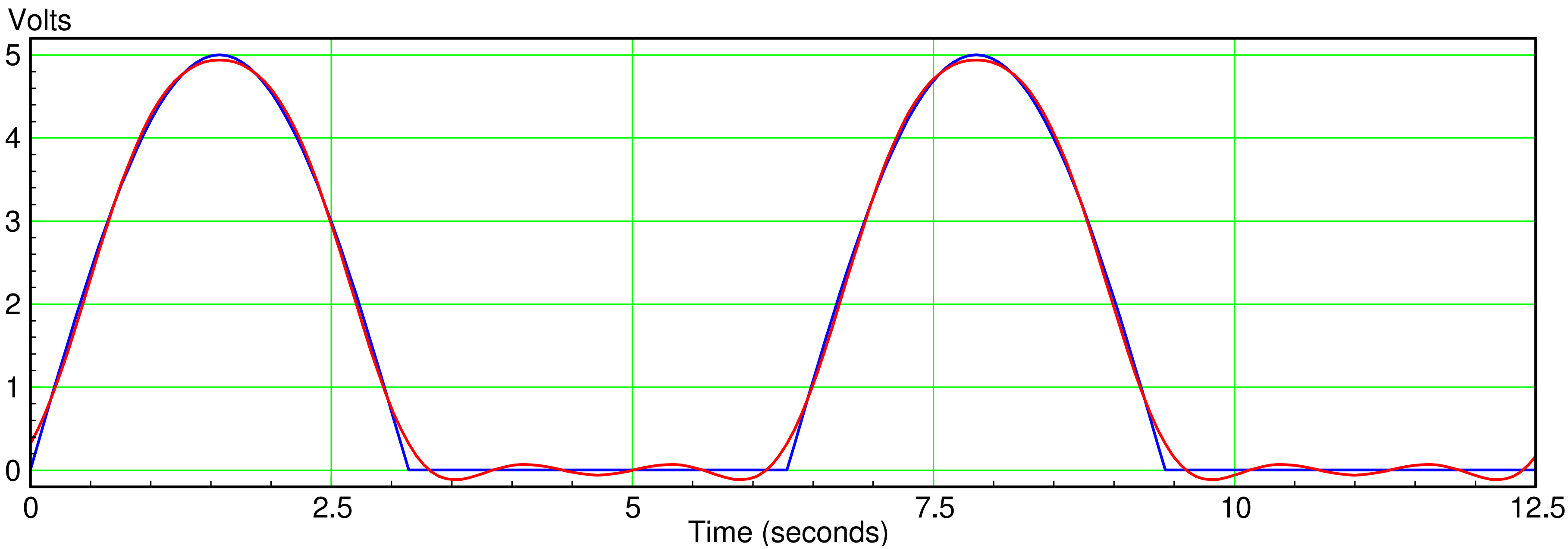


Half-Rectified Sine Wave

$$x(t) = x(t + 2\pi)$$

$$x(t) = \begin{cases} 5\sin(t) & \sin(t) > 0 \\ 0 & otherwise \end{cases}$$

$$x(t) \approx 1.591 + 2.5\sin(t) - 1.061\cos(2t) - 0.212\cos(4t) + \ldots$$



Summary:

If a waveform is periodic in time T,

It can be expressed as a sum of sine waves.

This allows us to use superposition to analyze the circuit for the given input without resorting to approximations.