Phasor Voltages

EE 206 Circuits I

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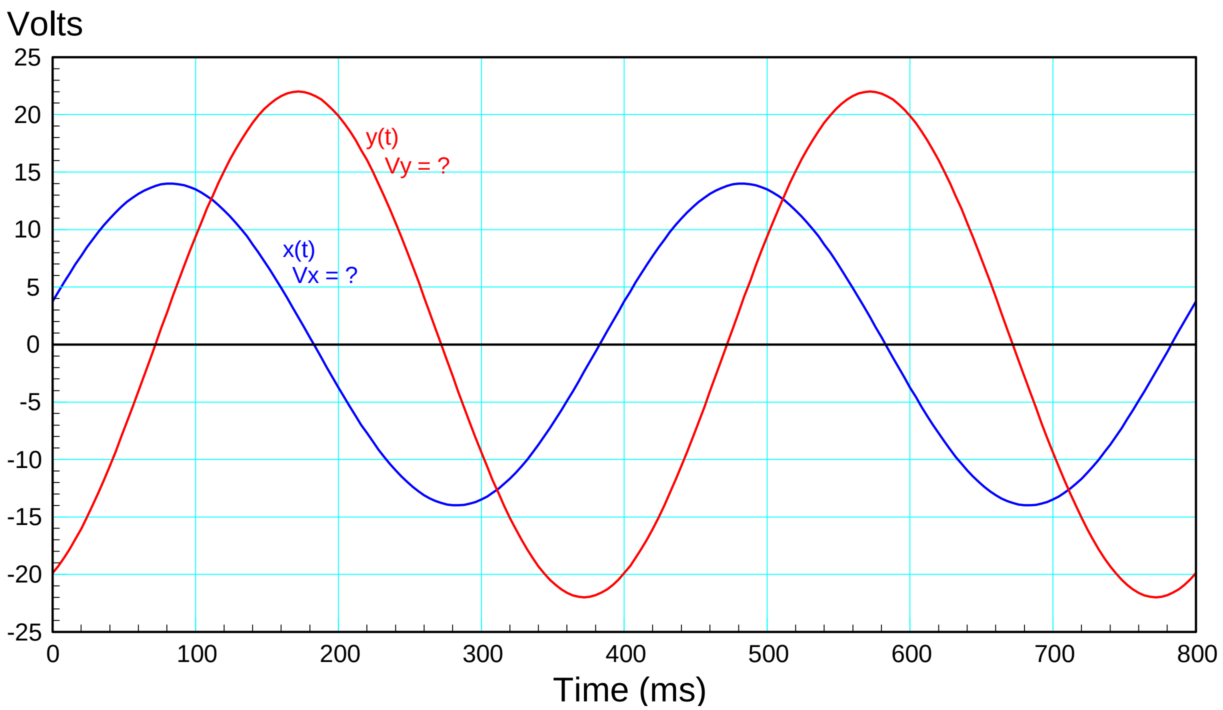
03/18/20

Please visit Bison Academy for corresponding lecture notes, homework sets, and solutions

Objective:

- Represent a sinusoid with a single complex number
- Express a sinusoid as seen on an oscilloscope as a complex number (a phasor)
- Determine the gain of a system from it's oscilloscope traces



Phasor Voltages:

A generic sinusoid at frequency w can be written as

$$x(t) = a\cos(\omega t) + b\sin(\omega t)$$

or

$$x(t) = r\cos(\omega t + \theta).$$

Note that to represent a sine wave, two terms are needed:

- The sine and cosine coefficients (termed rectangular form), or
- The amplitude (r) and phase shift (θ) (termed polar form).

Complex numbers can do that. The complex number representation for a sine wave is termed *it's phasor representation.*

Euler's Identity

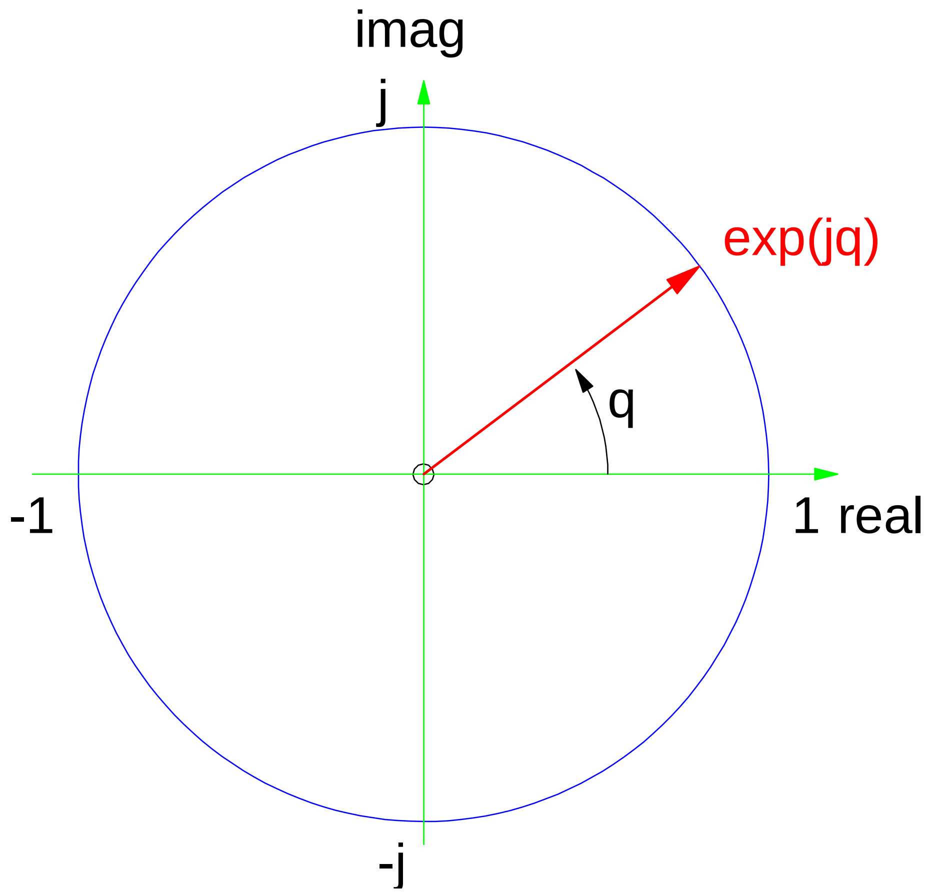
The heart of phasor representation is Euler's identity:

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

If you take the real part, you get cosine

$$real(e^{j\omega t}) = \cos(\omega t)$$

Hence, the phasor representation of cosine is 1.



Phasor Voltages: Rectangular Form

If you multiply by a complex number and take the real part, you get both sine and cosine:

$$(a + jb) \cdot e^{j\omega t} = (a + jb) \cdot (\cos(\omega t) + j \sin(\omega t))$$

$$= (a \cos(\omega t) - b \sin(\omega t)) + j(\cdots)$$

$$real((a + jb) \cdot e^{j\omega t}) = a \cos(\omega t) - b \sin(\omega t)$$

$$\boxed{a + jb \Leftrightarrow a \cos(\omega t) - b \sin(\omega t)}$$

Phasor Voltages: Polar Form

Similarly, if you multiply a complex exponential with a complex number in polar form, you get a cosine with a phase shift:

$$(r\angle\theta)\cdot e^{j\omega t} = (r\cdot e^{j\theta})\cdot e^{j\omega t}$$

$$= r\cdot e^{j(\omega t+\theta)}$$

$$= r(\cos(\omega t+\theta) + j\sin(\omega t+\theta))$$

$$real((r\angle\theta)\cdot e^{j\omega t}) = r\cdot\cos(\omega t+\theta)$$

$$\boxed{r\angle\theta \Leftrightarrow r\cdot\cos(\omega t+\theta)}$$

Phasor Domain vs. Time Domain

Frequency is understood (not written) when using phasors

- Capital letters donate phasor-domain
- Lower case letters donate time-domain

Phasor Domain	Time Domain
$V = 3 - j8$	$v(t) = 3\cos(20t) + 8\sin(20t)$
$V = 8\angle -23^0$	$v(t) = 8\cos(20t - 23^0)$

Addition and Subtraction of Voltages

Phasor Domain

$$V_1 = 3 - j8$$

$$V_2 = 2 + j6$$

$$V_3 = V_1 + V_2$$

$$V_3 = 5 - j2$$

Time Domain

$$v_1 = 3\cos(20t) + 8\sin(20t)$$

$$v_2 = 2\cos(20t) - 6\sin(20t)$$

$$v_3 = v_1 + v_2$$

$$v_3 = 5\cos(20t) + 2\sin(20t)$$

Addition in Polar Form

This also works in polar form (use a calculator):

Phasor Domain	Time Domain
$V_1 = 7\angle 15^0$	$v_1 = 7\cos(20t - 15^0)$
$V_2 = 9\angle 67^0$	$v_2 = 9\cos(20t + 67^0)$
$V_3 = V_1 + V_2$	$v_3 = v_1 + v_2$
$V_3 = 3.245 - j10.096$	

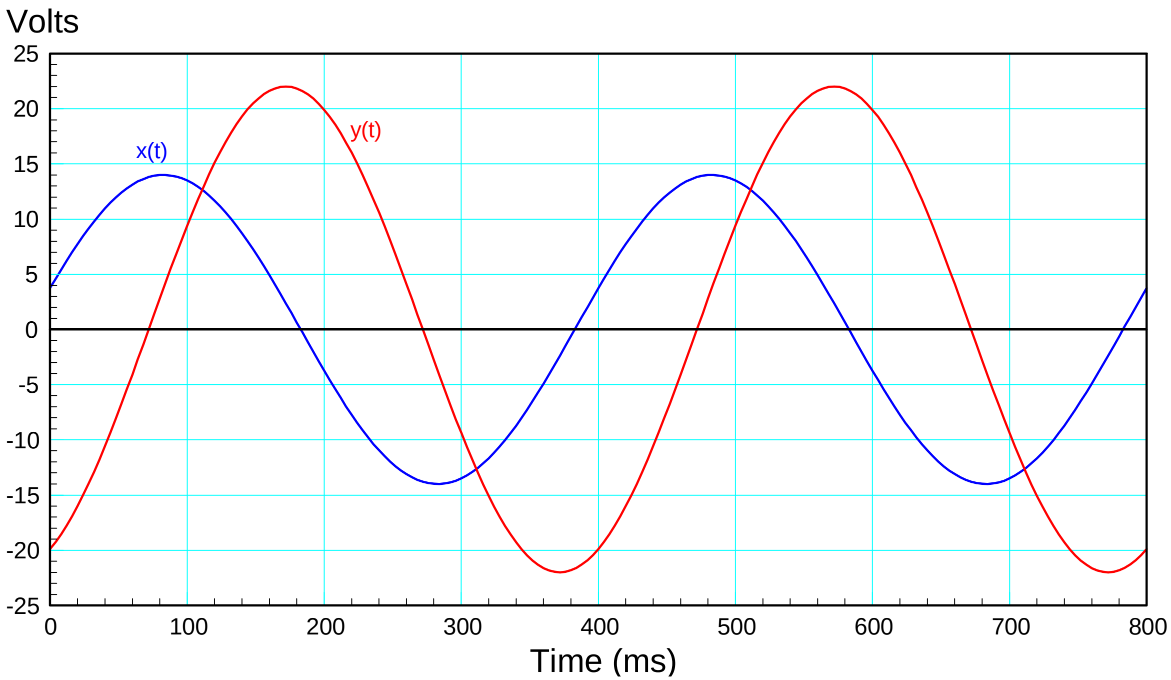
$$v_3 = 3.245\cos(20t) + 10.096\sin(20t)$$

note: V3 is found using a calculator which does complex numbers

Phasor Voltages: Experimental

In lab, you normally express a voltage in polar form. For example, determine the following from the following singnal from an oscilloscope:

- The frequency, and
- The phasor representation of X and Y



Frequency:

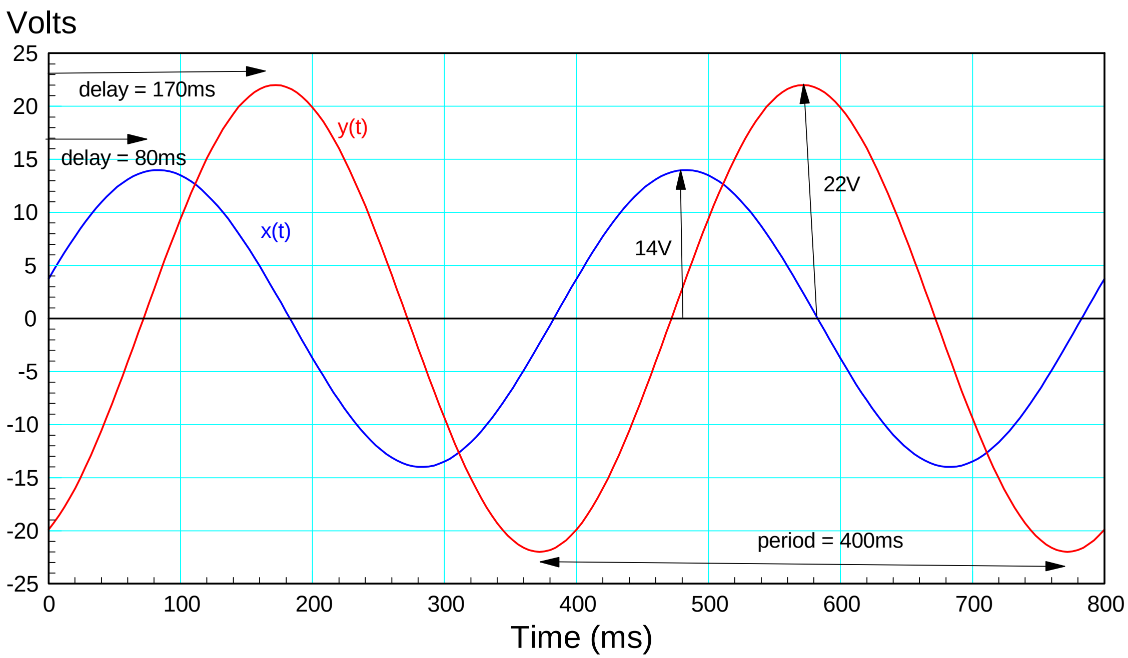
Frequency is defined as cycles per second or one over the period.

$$f = \frac{1}{T}\, hz$$

One cycle takes 400ms, so the frequency is

$$f = \frac{\text{one cycle}}{400ms} = 2.5hz$$

$$\omega = 2\pi f = 5\pi\, \frac{rad}{\text{sec}}$$

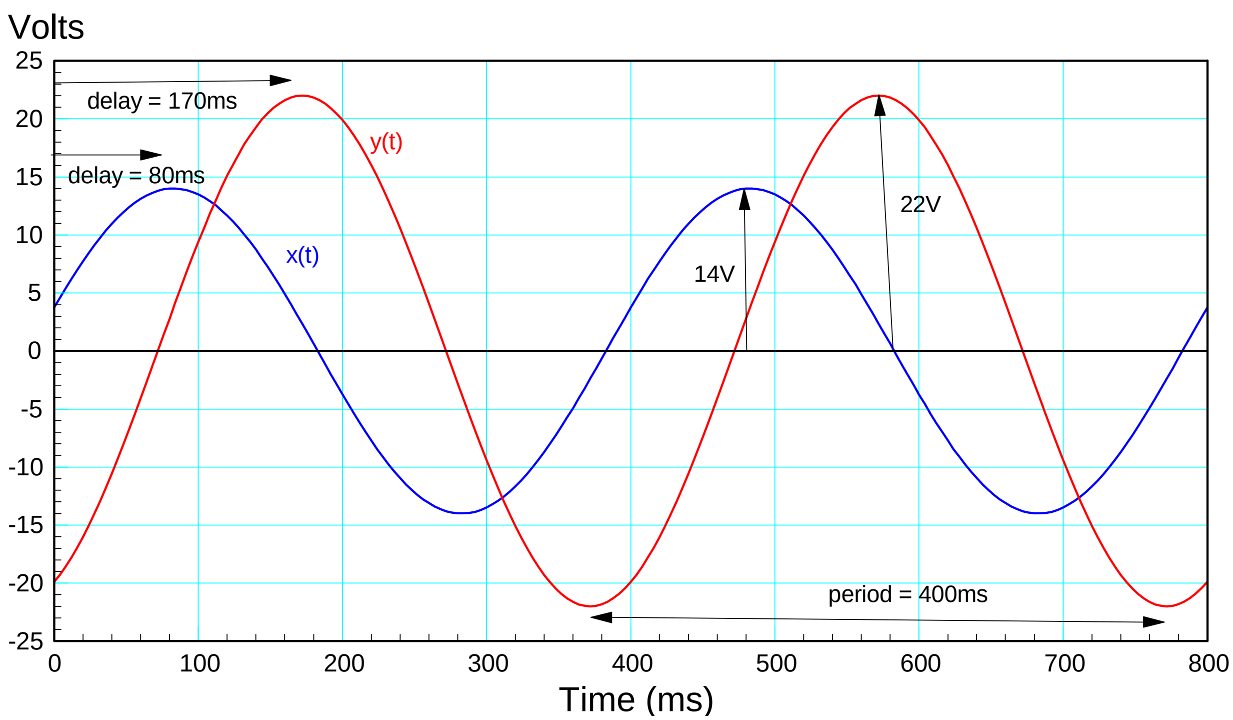


Peak Voltage:

The voltage from the average to the peak (Vp) is the amplitude:

|X| = 14V

|Y| = 22V



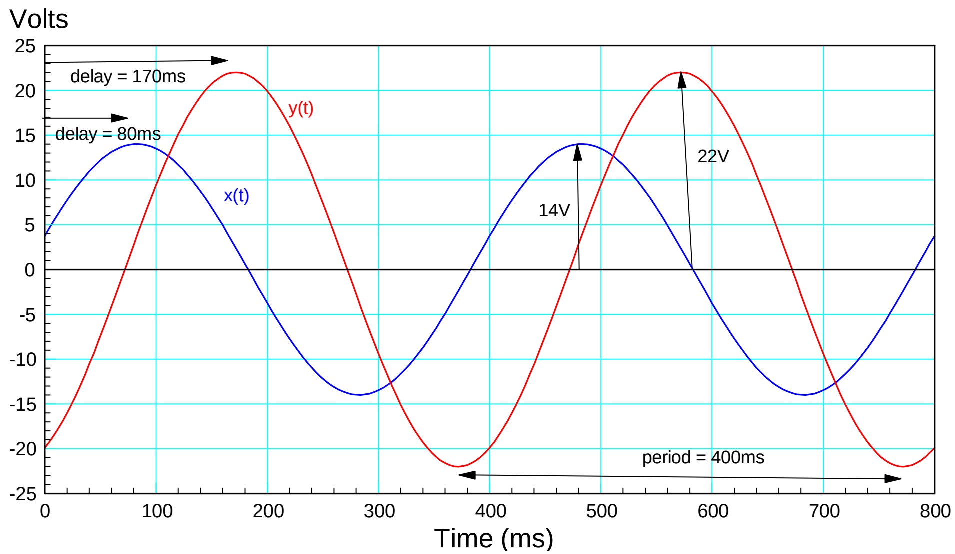
Phase Shift:

The delay is the phase shift (delay corresponds to a negative angle)

$$\theta = -\left(\frac{\text{delay}}{\text{period}}\right) \cdot 360^0$$

$$\theta_x = -\left(\frac{80ms}{400ms}\right) \cdot 360^0 = -72^0$$

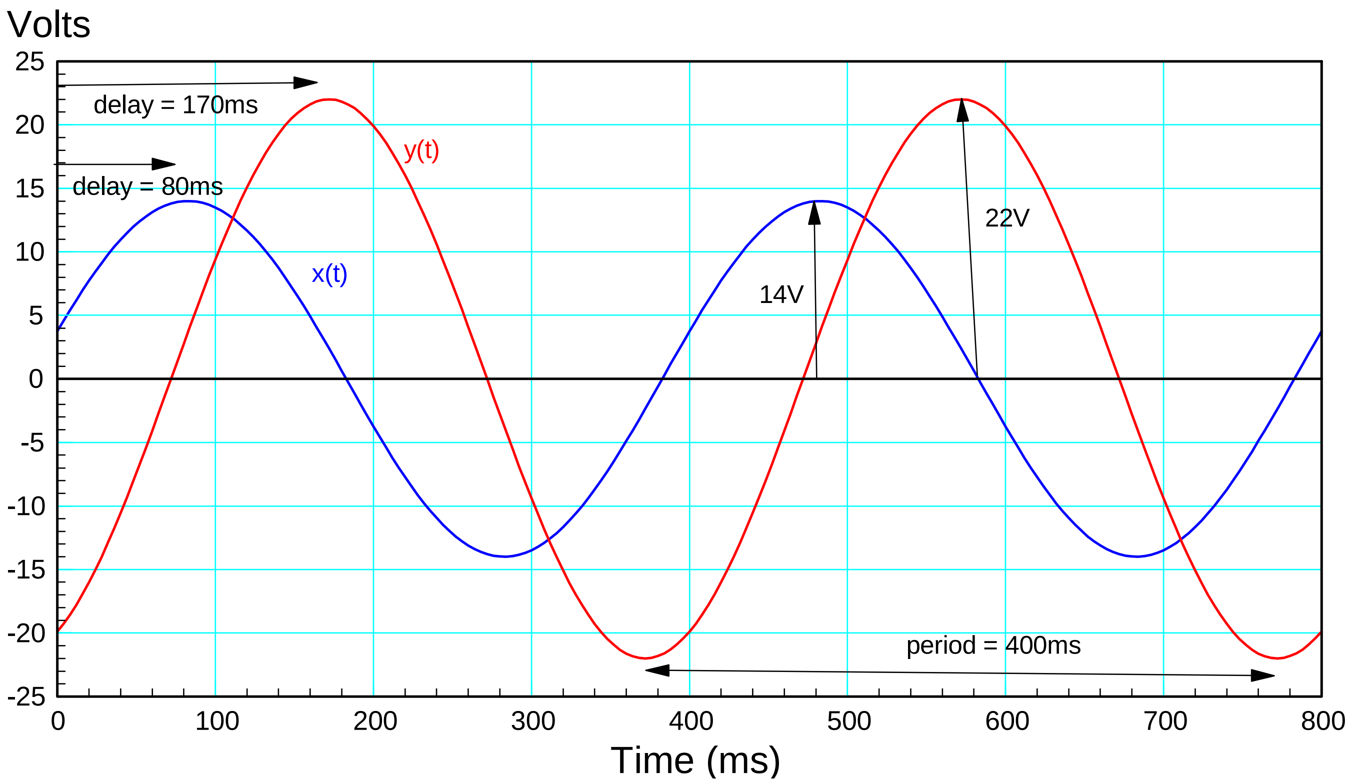
$$\theta_y = -\left(\frac{170ms}{400ms}\right) \cdot 360^0 = -153^0$$



Result:

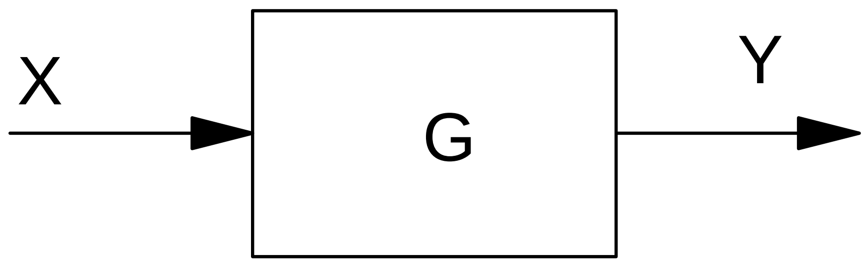
$$X = 14\angle -72^0$$

$$Y = 22\angle -153^0$$



Gain from X to Y:

A common problem with circuit analysis is to determine the gain of a circuit at a given frequency:



Gain is output / input

$$Y = G \cdot X$$

$$G = \frac{Y}{X}$$

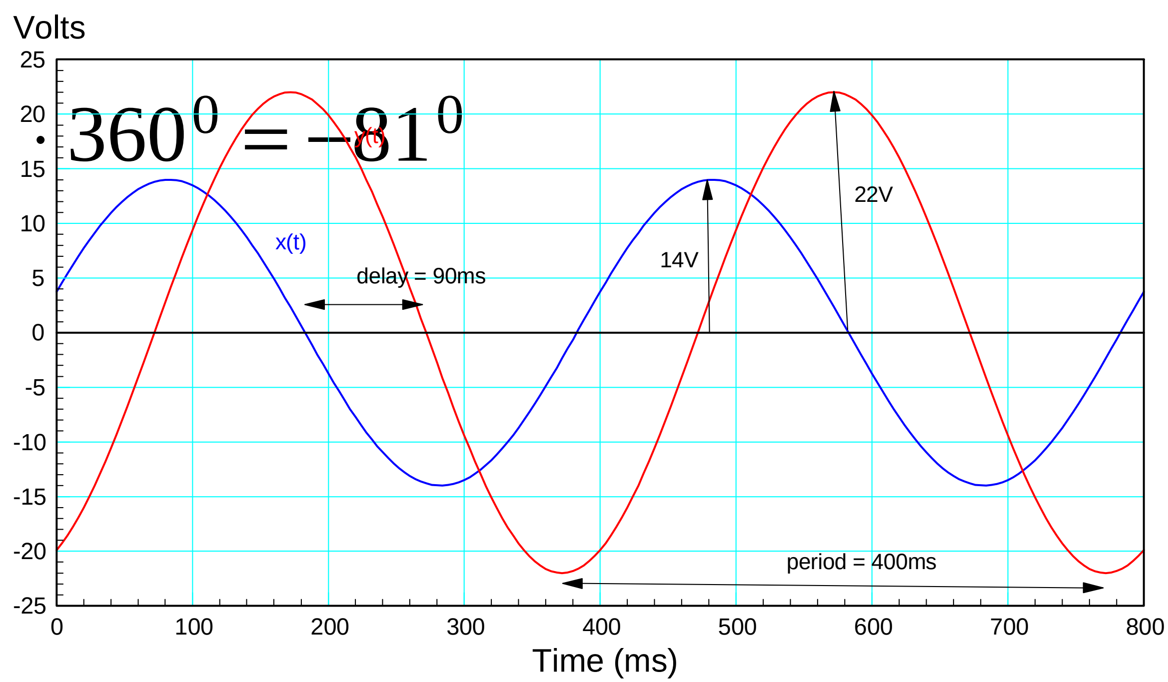
Gain Computations:

- The amplitude is the ratio: |Y| / |X|
- The phase is the difference: $\theta_g = \theta_y - \theta_x$

Similarly, with the previous data

$$|G| = \frac{22V}{14V} = 1.571$$

$$\angle G = -\left(\frac{\text{90ms delay X to Y}}{\text{400ms period}}\right) \bullet 360^0 = -81^0$$



The gain of this filter is

$$G = 1.571\angle -81^0$$