The p-Operator

Any time you try to find the voltages and currents for a circuit with inductors and capacitors, you are trying to solve a differential equation. The methods used in Calculus I- III are rather tedious for this purpose: there has to be a better way.

One tool to speed up the analysis of circuits with inductors and capacitors is to

- Use the p-operator (EE 206), or
- LaPlace Transforms (coming soon in Circuits II)

The idea is as follows.

Resistor circuits are fairly easy to analyze with the fundamental relationship

$$V = IR$$

If we can express the relationship for inductors and capacitors in the same way, circuit analysis with inductors and capacitors will be greatly simplified.

The basic equation for an inductor is:

$$v = L\frac{di}{dt}$$

Assume that the current is of the form

$$i(t) = a \cdot e^{pt}$$

Then

$$v(t) = L\frac{d}{dt}(ae^{pt})$$

$$v(t) = Lpae^{pt}$$

$$v(t) = Lp \cdot i(t)$$

This means that inductors look like resistors with a resistance of

$$Z_L = Lp$$

The basic equation for a capacitor is

$$i(t) = C\frac{dv}{dt}$$

Assume the voltage is of the form

$$v(t) = a \cdot e^{pt}$$

then

$$i(t) = \frac{d}{dt}(ae^{pt})$$

$$i(t) = Cp \cdot e^{pt}$$

$$i(t) = Cp \cdot v(t)$$

or

$$v(t) = \left(\frac{1}{Cp}\right) i(t)$$

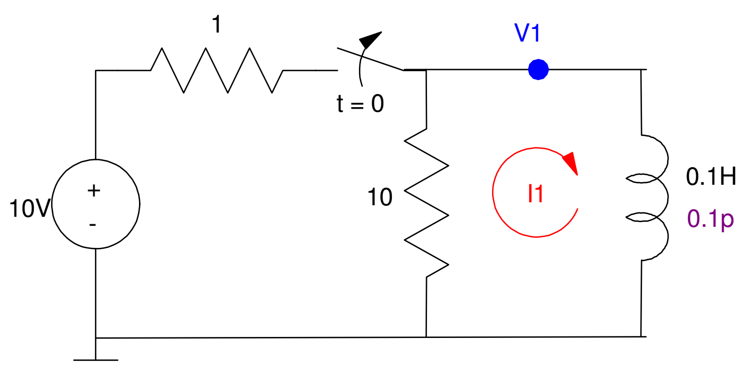
This means that capacitors look like resistors with an impedance of

$$Z_c = \left(\frac{1}{Cp}\right)$$

What the p-operator does is it changes using calculus to solve circuits problems to using algebra - with the assumption that algebra is easier than calculus. Solving the same problems as yesterday using the p-opeator hopefully gives the same answers with a lot less work.

Natural Response of an RL Circuit

Find the voltage v1(t). Assume the switch opens up at t=0



The initial condition (just prior to t = 0)

$$i_1(0) = \left(\frac{10V}{1\Omega}\right) = 10A$$

The loop equation becomes

$$10I_1 + 0.1p \cdot I_1 = 0$$

$$(10 + 0.1p)I_1 = 0$$

meaning that

$$p = -100$$

and

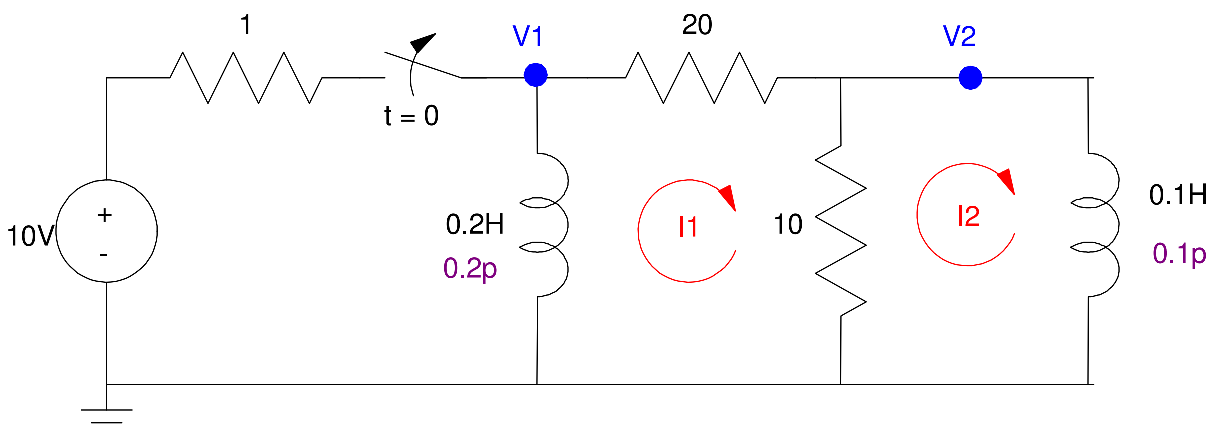
$$i_1(t) = ae^{-100t}$$

Plugging in the initial condition results in

$$i_1(t) = 10e^{-100t} \qquad t > 0$$

Natural Response of a 2-stage RL Filter

Find the voltages, v1(t) and v2(t). Assume the switch has been closed for a long time at opens at t = 0.



The initial conditions are

$$i_1(0) = \frac{10V}{1\Omega} = 10A$$

$$i_2(0) = 0$$

Using the p-operator, the loop equations become

$$0.2pI_1 + 20I_1 + 10(I_1 - I_2) = 0$$

$$10(I_2 - I_1) + 0.1pI_2 = 0$$

To solve, group terms

$$(0.2p + 30)I_1 - 10I_2 = 0 \qquad * \; 10$$

$$-10I_1 + (0.1p + 10)I_2 = 0 \qquad * \; (0.2p + 30)$$

Solve using Gauss elimination

$$-100I_2 + (0.1p+10)(0.2p+30)I_2 = 0$$

$$(0.02p^2 + 5p + 200)I_2 = 0$$

Factor

$$0.02(p+50)(p+200)I_2 = 0$$

meaning

p = -50 or -200

meaning

$$i_2(t) = ae^{-50t} + be^{-200t}$$

This also tells you what i1(t) is. From the second differential equation

$$10(i_2 - i_1) + 0.1\frac{di_2}{dt} = 0$$

you get

$$i_1 = 0.01\frac{di_2}{dt} + i_2$$

Plugging in the initial conditions

(1) $i_2(0) = 0 = a + b$

(2) $i_1(0) = 10 = 0.01(-50a - 200b) + (a+b)$

$$10 = 0.5a - b$$

Solving

$$a = 6.667$$

$$b = -6.667$$

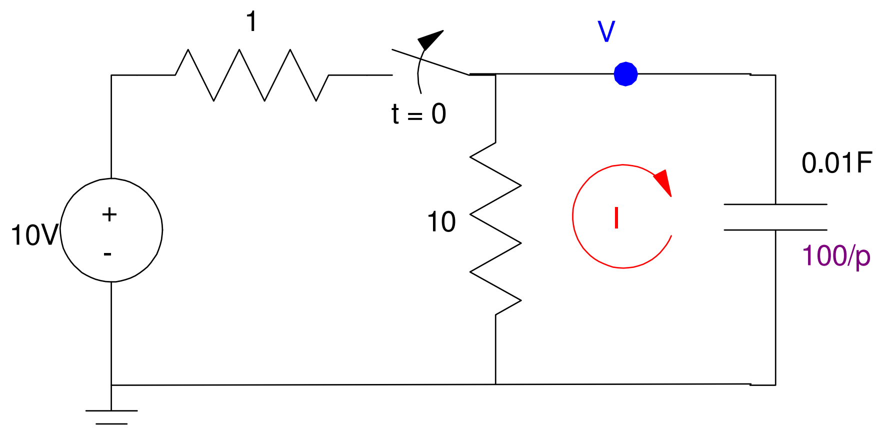
and

$$\boxed{i_2(t) = 6.667e^{-50t} - 6.667e^{-200t}} \qquad t > 0$$

which is the same answer as we got before.

RC Circuits and the operator

Assume for the following circuit the switch opens at t=0. Find V(t)



Write the voltage node equation at V:

$$\frac{V}{10} + \frac{V}{100/p} = 0$$

$$(0.1 + 0.01p)V = 0$$

$$p = -10$$

This tells you that

$$v(t) = a \cdot e^{-10t}$$

Plugging in the initial condition

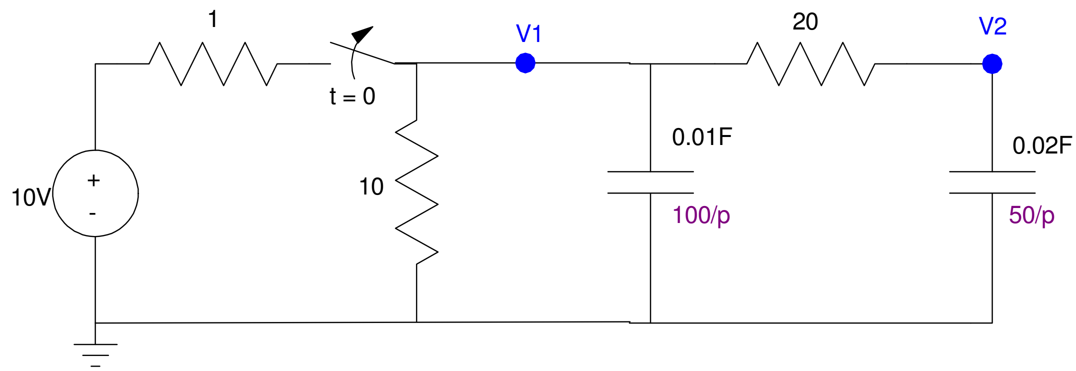
$$v(0) = \left(\frac{10}{10+1}\right) \cdot 10V = 9.09V$$

results in

$$v(t) = 9.09 \cdot e^{-10t} \qquad t > 0$$

Same answer as before

2-Stage RC Filter



The initial conditions are

$$v_1(0) = \left(\frac{10}{10+1}\right) 10V = 9.09V$$

$$v_2(0) = v_1(0) = 9.09V$$

Using the p-operator, the voltage node equations become

$$\left(\frac{V_1}{10}\right) + \left(\frac{V_1}{100/p}\right) + \left(\frac{V_1 - V_2}{20}\right) = 0$$

$$\left(\frac{V_2 - V_1}{20}\right) + \left(\frac{V_2}{50/p}\right) = 0$$

Doing some algebra...

$$(p+15)V_1 - 5V_2 = 0 \qquad * 2.5$$

$$-2.5V_1 + (p+2.5)V_2 = 0 \qquad * (p+15)$$

Solving for V2

$$((p+2.5)(p+15) - 12.5)V_2 = 0$$

$$(p^2 + 17.5p + 25)V_2 = 0$$

$$(p+1.569)(p+15.931)V_2 = 0$$

meaning

p = {-1.569, -15.931}

and

$$v_2(t) = a \cdot e^{-1.569t} + b \cdot e^{-15.931t}$$

Plugging in the initial conditions

$$v_2(0) = 9.09 = a + b$$

and

$$V_1 = (0.4p + 1)V_2$$

The operator 'p' means 'the derivative of', or

$$v_1(0) = 9.02 = 0.4(-1.569a = 15.93n) + (a + b)$$

Solving 2 equations for 2 unknowns results in

$$a = 10.083$$

$$b = -0.993$$

and

$$\boxed{v_2(t) = 10.083e^{-1.569} - 0.993e^{-15.931t}} \quad t > 0$$

which is what we got before.