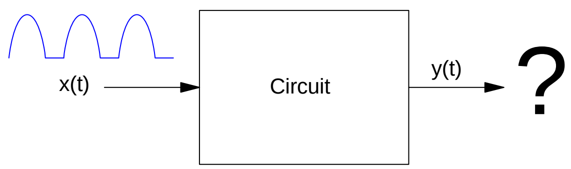
Fourier Transform

Superposition allows you to analyze circuits with multiple sinusoidal inputs. If this is the case

- Treat the problem as N separate problems, each with a single sinusoidal input.
- Solve each of the N problems separately using phasor analysis
- Add up all of the answers to get the total output.

Suppose your circuit has an input that *isn't* a sum of sinusoids.

- One solution is to approximate the input with two sine wave (what we did last lecture)
- A second solution is to define the input in terms of sine waves (this lecture)



Fourier Transform: How to deal with periodic inputs which are *not* sinusoids

The Fourier Transform is a tool which allows you to take signal which is periodic in time T

$$x(t) = x(t+T)$$

and express it as an infinite series

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

where

$$\omega_0 = \frac{2\pi}{T}$$

Equals is a very powerful symbol: it states that both sides of the equation are equivalent

- The left side is a periodic function which *isn't* expressed in terms of sine waves
- The right side is a periodic function which *is* expressed in terms of sine waves.

We like the right side: it allows us to use phasor analysis to solve any circuit.

What the Fourier Transform says is, going right to left

If you add up a bunch of signals which are periodic in time T, the result is also periodic in time T

Duh. That's not the least bit surprising. Going left to right, however, is much more significant

If a signal is periodic and is not a sine wave, it is made up of sine waves which are harmonics of the fundamental.

To determine the Fourier coefficient, there are several methods which you'll cover in Circuits II and Signals and Systems. A numerical solution using Matlab is

$$a_0 = mean(x)$$

$$a_n = 2 \cdot mean(x \cdot \cos(n\omega_0 t))$$

$$b_n = 2 \cdot mean(x \cdot \sin(n\omega_0 t))$$

Proof: All sine waves are orthogonal. The DC term is

$$a_0 = mean(x)$$

$$a_0 = mean(a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))$$

$$a_0 = mean(a_0) + mean(a_1 \cos(\omega_0 t)) + mean(a_2 \cos(2\omega_0 t)) + ...$$

The mean of a sine wave is zero

$$a_0 = a_0 + 0 + 0 + ...$$

Term a1:

$$a_1 = 2mean(x \cdot \cos(\omega_0 t))$$

$$a_1 = 2mean((a_0 + a_1 \cos(\omega_0 t) + b_1 \sin(\omega_0 t) + ...) \cdot \cos(\omega_0 t))$$

$$a_1 = 2mean(a_0 \cdot \cos(\omega_0 t)) + 2mean(a_1 \cos^2(\omega_0 t)) + 2mean(b_1 \sin(\omega_0 t)\cos(\omega_0 t)) + ...$$

The mean of a sine wave is zero. The mean of $\sin^2(t)$ is 1/2

$$a_1 = 0 + 2 \cdot \frac{a_1}{2} + 0 + ...$$

$$a_1 = a_1$$

etc.

Time Scaling: The period of the periodic waveform has no impact on the Fourier coefficients. If you do a change in variable (i.e. change the time scaling) you can always make the period equal to 2π. This is convenient since it makes the fundamental frequency 1 rad/sec

$$T = 2\pi$$

(at some definition of time)

$$\omega_0 = \frac{2\pi}{T} = 1$$

Some Common Fourier Transforms

Sine Wave: Express

$$x(t) = 3\cos(5t)$$

in terms of it's Fourier series.

Answer: You're already there...

$$x(t) = 3\cos(5t)$$

Square Wave: Express a 0V / 5V 5 rad/sec square wave in terms of its Fourier Transform

$$x(t) = \begin{cases} 5V & \cos(5t) > 0 \\ 0V & otherwise \end{cases}$$

The fundamental frequency is 5 rad/sec. The period is

$$T = \frac{2\pi}{\omega_0} = 1.257 \text{ seconds}$$

In Matlab, you can find the Fourier coefficients

```
Wo = 5;
T = 2*pi / Wo;
t = [0:0.0001:1]' * T;

x = 5 * (cos(5*t) > 0);

a0 = mean(x)

 a0  = 2.50025

a1 = 2*mean(x .* cos(Wo*t))

 a1  = 3.1837804

b1 = 2*mean(x .* sin(Wo*t));

a2 = 2*mean(x .* cos(2*Wo*t));
b2 = 2*mean(x .* sin(2*Wo*t));

a3 = 2*mean(x .* cos(3*Wo*t));
b3 = 2*mean(x .* sin(3*Wo*t));

a4 = 2*mean(x .* cos(4*Wo*t));
b4 = 2*mean(x .* sin(4*Wo*t));

a5 = 2*mean(x .* cos(5*Wo*t));
b5 = 2*mean(x .* sin(5*Wo*t));
```

This results in

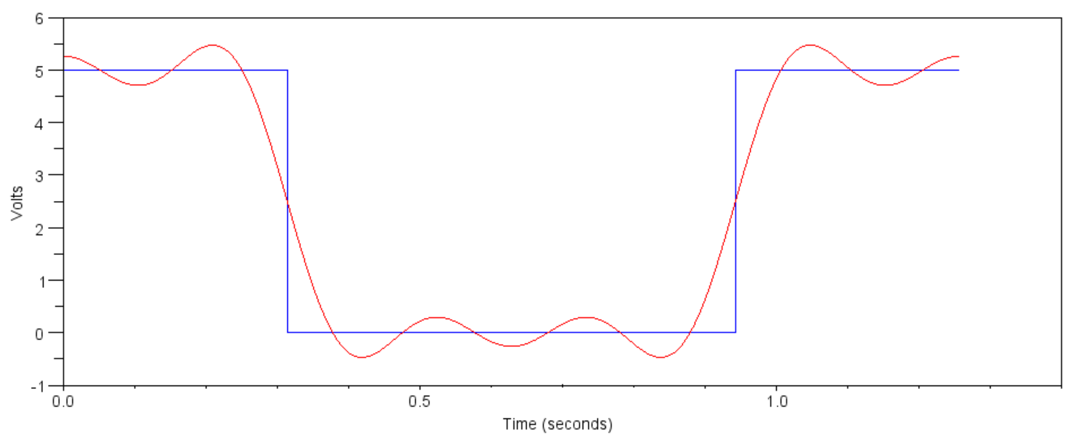
n	0	1	2	3	4	5
a_n	2.5	3.18	0	1.06	0	0.64
b_n	0	0	0	0	0	0

What this means is

$$x(t) \approx 2.5 + 3.18\cos(5t) + 1.06\cos(15t) + 0.64\cos(25t) + \ldots$$

Comparing the original function to its Fourier approximation

```
y = a0 + a1*cos(5*t) + a3*cos(15*t) + a5*cos(25*t);
plot(t,x,'b',t,y,'r');
xlabel('Time (seconds)');
ylabel('Volts');
```



Square Wave (blue) and Fourier Approximation taken out to the 5th Harmonic (red)

The red line (the Fourier approximation) doesn't follow the blue line (x(t)) exactly. You need more terms to make the two match up exactly.

Triangle Wave: Find the Fourier transform for a triangle wave with a period of 2 seconds:

$$x(t) = x(t+2)$$

$$x(t) = \begin{cases} t & 0 < t < 1 \\ 2-t & 1 < t < 2 \end{cases}$$

Solution: The fundamental frequency is

$$T = 2$$

$$\omega_0 = \frac{2\pi}{T} = \pi$$

In Matlab:

```
T = 2;
t = [0:0.0001:1]' * T;
Wo = 2*pi/T;

x = t .* (t<1)  +  (2-t) .* (t>=1);
```

Now it's the same as before

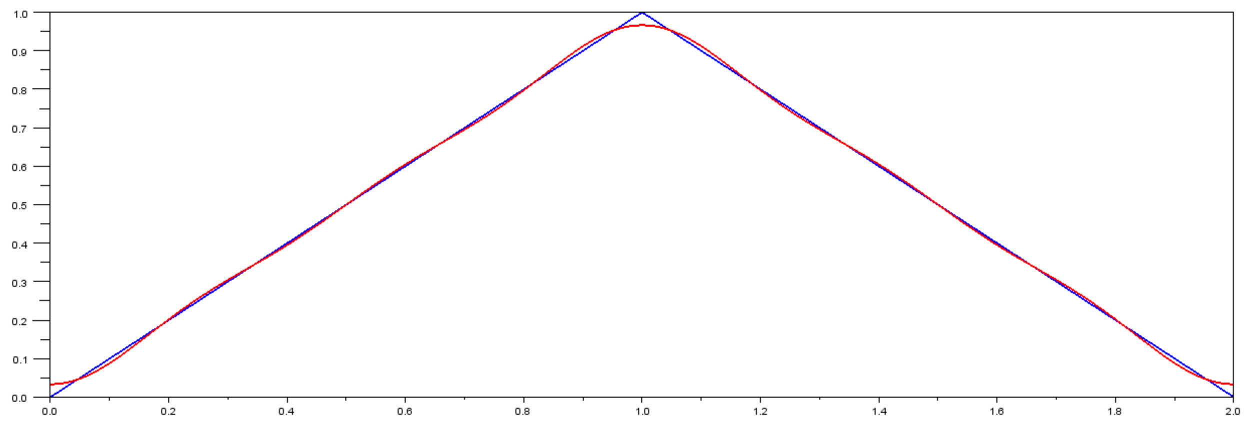
```
a0 = mean(x)

a1 = 2*mean(x .* cos(Wo*t))
a2 = 2*mean(x .* cos(2*Wo*t))
a3 = 2*mean(x .* cos(3*Wo*t))
a4 = 2*mean(x .* cos(4*Wo*t))
a5 = 2*mean(x .* cos(5*Wo*t))

b1 = 2*mean(x .* sin(Wo*t))
b2 = 2*mean(x .* sin(2*Wo*t))
b3 = 2*mean(x .* sin(3*Wo*t))
b4 = 2*mean(x .* sin(4*Wo*t))
b5 = 2*mean(x .* sin(5*Wo*t))
```

The result is

n	0	1	2	3	4	5
a_n	0.50	-0.405	0	-0.045	0	-0.016
b_n	0	0	0	0	0	0

Plotting the Fourier approximation taken out to the 5th harmonic vs. the triangle wave looks like the following:

```
y = a0 + a1*cos(Wo*t) + a2*cos(2*Wo*t) + a3*cos(3*Wo*t) + a4*cos(4*Wo*t) + a5*cos(5*Wo*t);
y = y + b1*sin(Wo*t) + b2*sin(2*Wo*t) + b3*sin(3*Wo*t) + b4*sin(4*Wo*t) + b5*sin(5*Wo*t);
plot(t,x,'b',t,y,'r');
```



Triangle Wave (blue) and Fourier Approximation taken out to the 5th Harmonic (red)

Half-Wave Rectified Sine Wave: Finally, determine the Fourier series approximation to a half-wave rectified sine wave:

$$x(t) = x(t + 2\pi)$$

$$x(t) = \begin{cases} 5\sin(t) & \sin(t) > 0 \\ 0 & otherwise \end{cases}$$

Solution: Same as before. The period and fundamental frequency are

$$T = 2\pi$$

$$\omega_0 = \frac{2\pi}{T} = 1$$

The Fourier Coefficients from Matlab are (same as before except for x(t))

```
T = 2*pi;
t = [0:0.0001:1]' * T;
Wo = 2*pi/T;
x = max(0, 5*sin(t));a0 = mean(x)

a0 = mean(x)
a1 = 2*mean(x .* cos(Wo*t))
a2 = 2*mean(x .* cos(2*Wo*t))
a3 = 2*mean(x .* cos(3*Wo*t))
a4 = 2*mean(x .* cos(4*Wo*t))
a5 = 2*mean(x .* cos(5*Wo*t))

b1 = 2*mean(x .* sin(Wo*t))
b2 = 2*mean(x .* sin(2*Wo*t))
b3 = 2*mean(x .* sin(3*Wo*t))
b4 = 2*mean(x .* sin(4*Wo*t))
b5 = 2*mean(x .* sin(5*Wo*t))
```

This results in

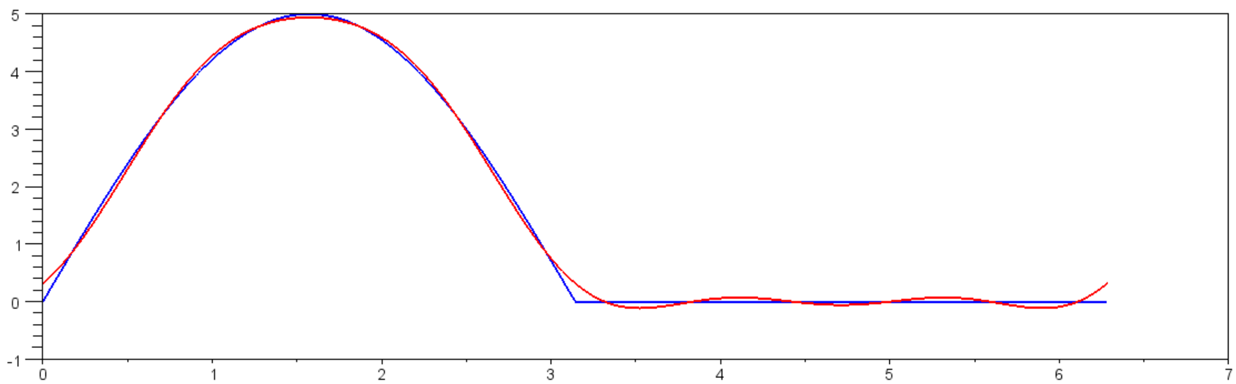
n	0	1	2	3	4	5
a_n	1.591	0	-1.061	0	-0.212	0
b_n	0	2.500	0	0	0	0

meaning

$$x(t) \approx 1.591 + 2.5\sin(t) - 1.061\cos(2t) - 0.212\cos(4t) + \ldots$$

Plotting x(t) against its Fourier series approximation taken out to the 5th harmonic looks like the following:

```
y = a0 + a1*cos(Wo*t) + a2*cos(2*Wo*t) + a3*cos(3*Wo*t) + a4*cos(4*Wo*t) + a5*cos(5*Wo*t);
y = y + b1*sin(Wo*t) + b2*sin(2*Wo*t) + b3*sin(3*Wo*t) + b4*sin(4*Wo*t) + b5*sin(5*Wo*t);
plot(t,x,'b',t,y,'r')
```



Half-Rectified Sine Wave (blue) and Fourier Series Approximation (red)

Like before, if you add more terms the Fourier Series approximation converges to the actual signal.