Voltage Nodes with Phasors

	VI relationship	Phasor Notation
Voltage	$v(t) = a\cos(\omega t) + b\sin(\omega t)$	$V = a - jb$
Resistor	$v = iR$	$Z_R = R$
Inductor	$v = L\frac{di}{dt}$	$Z_L = j\omega L$
Capacitor	$i = C\frac{dv}{dt}$	$Z_C = \frac{1}{j\omega C}$

Introduction

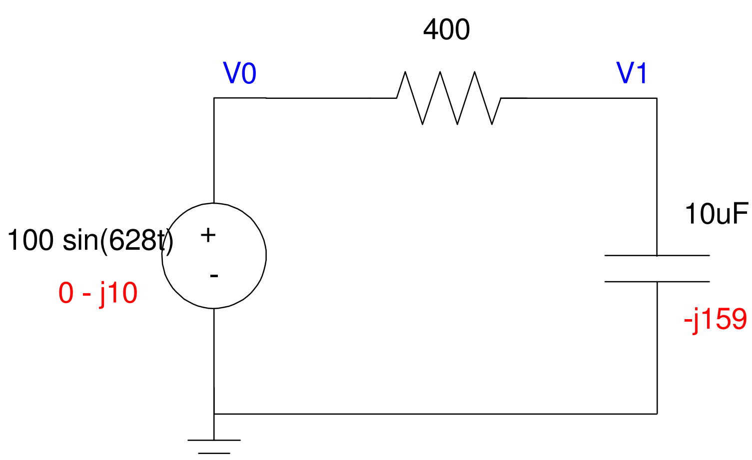
Kirchoff's Voltage Nodes states that the current from a node must sum to zero. This applies at DC and at AC. The only difference is for AC circuits

- You are dealing with complex numbers (i.e. phasors), and
- Both the real and complex part of the current must sum to zero.

Essentially, if you don't mind using complex numbers, voltage nodes for AC circuits is just like AC analysis for DC circuits.

Example 1: RC Circuit

Determine the node voltages for the following circuit:



Step 1: Replace the voltage source and capacitor with their phasor value:

$$\omega = 628 \ \frac{rad}{\sec}$$

$$10\sin(628t) \rightarrow 0 - j10$$

$$10\mu F \rightarrow \frac{1}{j\omega C} = -j159\Omega$$

Step 2: Write the voltage node equations just like we did at DC:

$$V_0 = 0 - j100$$

$$\left(\frac{V_1 - V_0}{400}\right) + \left(\frac{V_1}{-j159}\right) = 0$$

Solve:

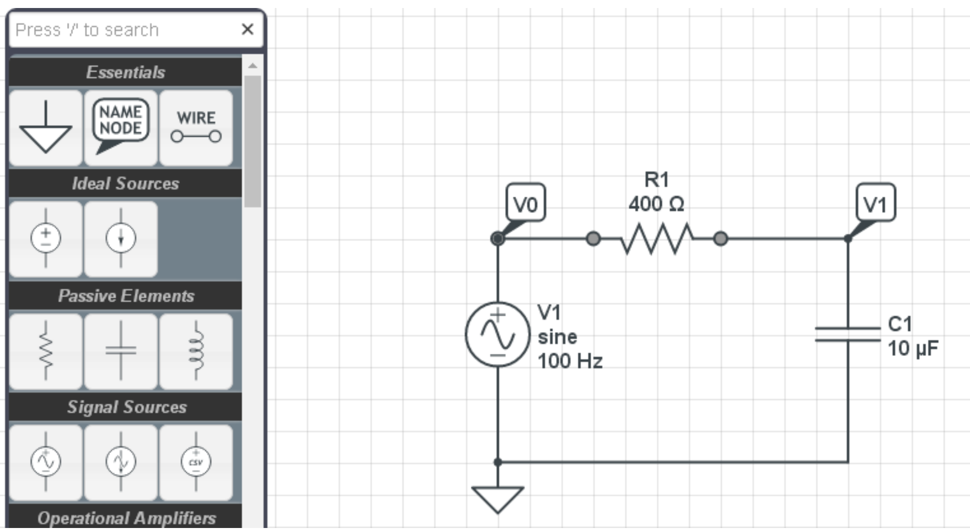
$$V_1 = \left(\frac{-j159}{-j159+400}\right)(0 - j100)$$

$$V_1 = 34.326 - j13.645$$

$$v_1(t) = 34.326\cos(628t) + 13.645\sin(628t)$$

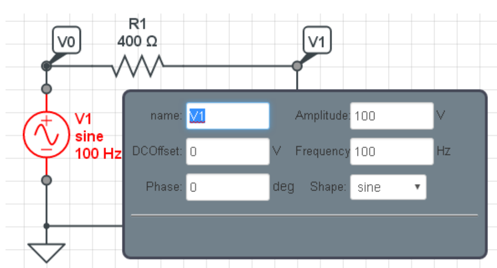
In CircuitLab (www.CircuitLab.com)

Input the circuit using drag and drop



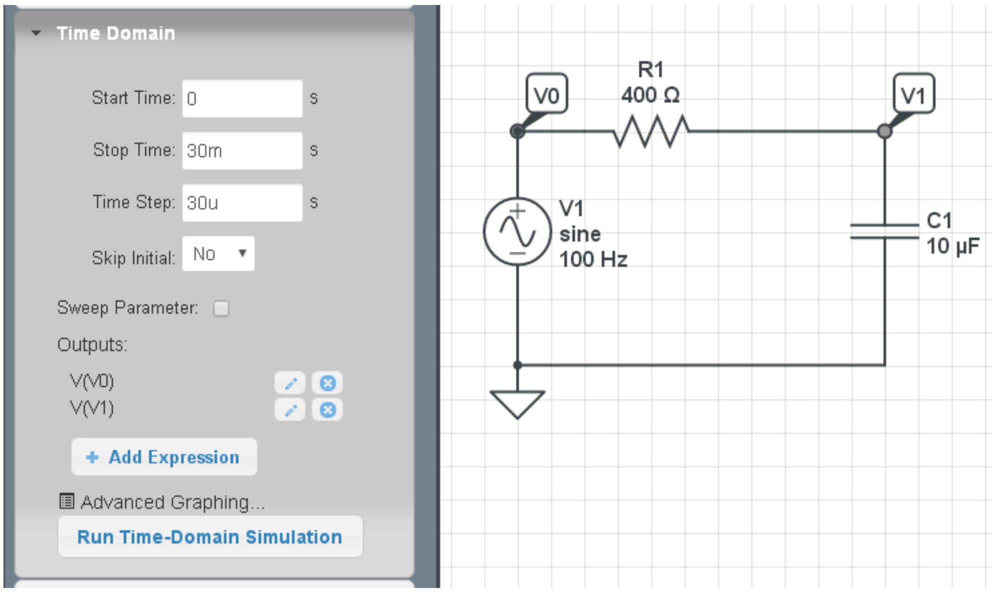
Make the input a sine wave with

- no DC offset
- 100V amplitude
- 100Hz (628 rad/sec)

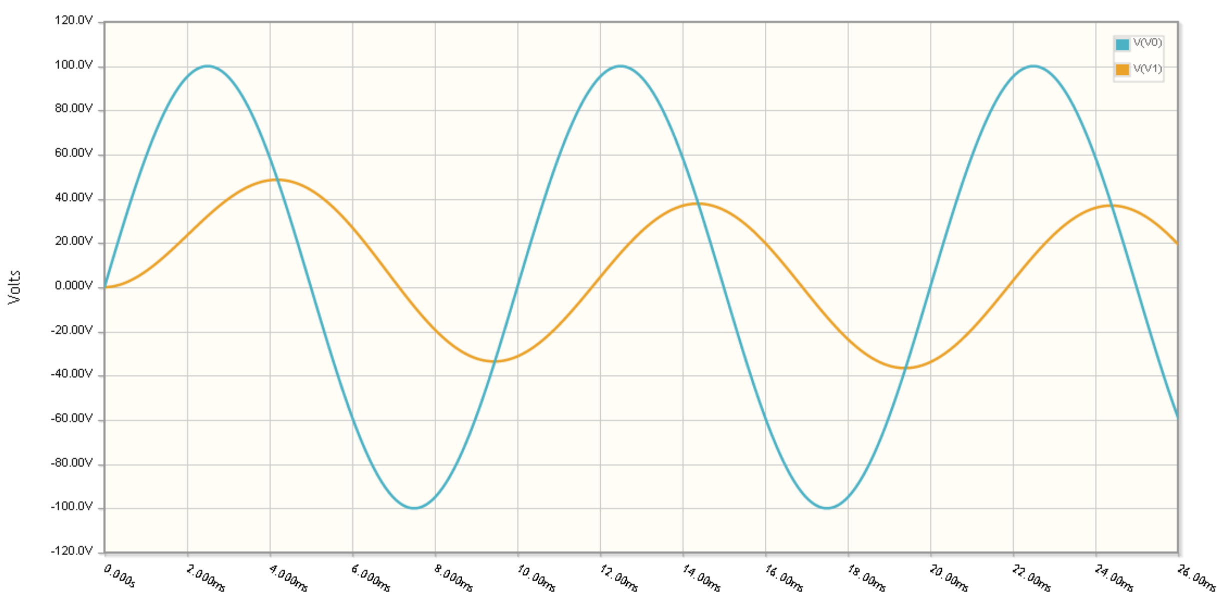


Run a transient response for

- 30ms (3 cycles)
- 30us step size (1000 points on the plot)



This results in a simulated input and output waveform:



Note from the CircuitLab plot, the output after a short transient is

- 36.91Vp
- The peak of V1 is delayed by 4.39ms from the zero crossing of V0 (the blue curve)

The zero crossing is used as a reference since V0 = 100 sin(628t) (V0 is zero at t=0). The phase shift is thus

$$\theta_1 = -\left(\frac{4.39ms\text{ delay}}{10\text{ms period}}\right)360^0$$

$$\theta_1 = -158^0$$

hence

$$V_1 = 37\angle -158^0$$

This matches the polar form of V2(t)

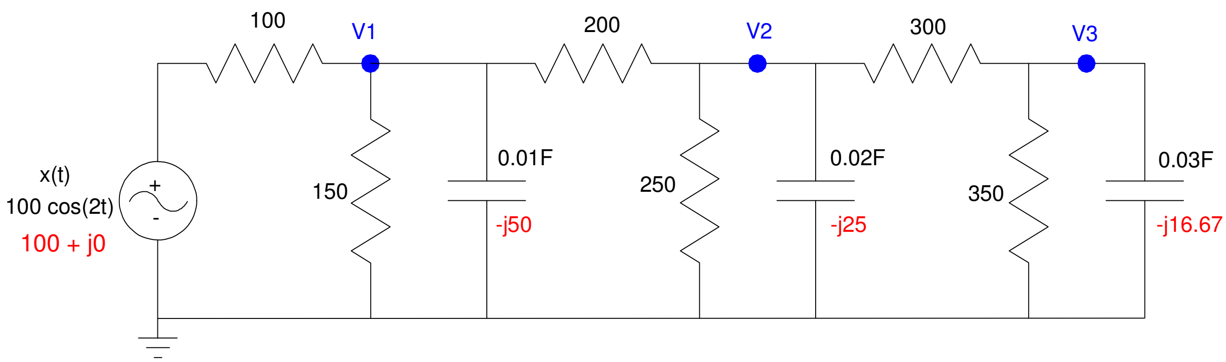
$$V_1 = 34.326 - j13.645$$

$$V_1 = 36.939\angle -158.3^0$$

Example 2: 3-Stage RC Circuit

Find the voltages for the following circuit when the input is

x(t) = 100 cos(2t)



Step 1: Change the capacitors and voltages to their phasor representation (shown in red)

$$\omega = 2$$

x(t)

$$X = 100 + j0$$

0.01F:

$$Z_c = \frac{1}{j\omega C} = -j50\Omega$$

0.02F:

$$Z_c = \frac{1}{j\omega C} = -j25\Omega$$

0.03F:

$$Z_c = \frac{1}{j\omega C} = -j16.67\Omega$$

Step 2: Write N equations for N unknowns

V1: $\left(\frac{V_1 - X}{100}\right) + \left(\frac{V_1}{150}\right) + \left(\frac{V_1}{-j50}\right) + \left(\frac{V_1 - V_2}{200}\right) = 0$

V2: $\left(\frac{V_2 - V_1}{200}\right) + \left(\frac{V_2}{250}\right) + \left(\frac{V_2}{-j25}\right) + \left(\frac{V_2 - V_3}{300}\right) = 0$

V3: $\left(\frac{V_3 - V_2}{300}\right) + \left(\frac{V_3}{350}\right) + \left(\frac{V_3}{-j16.67}\right) = 0$

Step 3: Solve.

First, group terms

$$\left(\frac{1}{100}+\frac{1}{150}+\frac{1}{-j50}+\frac{1}{200}\right)V_1+\left(\frac{-1}{200}\right)V_2=\left(\frac{1}{100}\right)X$$

$$\left(\frac{-1}{200}\right)V_1+\left(\frac{1}{200}+\frac{1}{250}+\frac{1}{-j25}+\frac{1}{300}\right)V_2+\left(\frac{-1}{300}\right)V_3=0$$

$$\left(\frac{-1}{300}\right)V_2+\left(\frac{1}{300}+\frac{1}{350}+\frac{1}{-j16.67}\right)V_3=0$$

Place in matrix form

$$\begin{bmatrix} \left(\frac{1}{100}+\frac{1}{150}+\frac{1}{-j50}+\frac{1}{200}\right) & \left(\frac{-1}{200}\right) & 0 \\ \left(\frac{-1}{200}\right) & \left(\frac{1}{200}+\frac{1}{250}+\frac{1}{-j25}+\frac{1}{300}\right) & \left(\frac{-1}{300}\right) \\ 0 & \left(\frac{-1}{300}\right) & \left(\frac{1}{300}+\frac{1}{350}+\frac{1}{-j16.67}\right) \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}=\begin{bmatrix} \left(\frac{1}{100}\right) \\ 0 \\ 0 \end{bmatrix}X$$

Put into MATLAB and solve

```
a11 = 1/100 + 1/150 - 1/(j*50) + 1/200;
a12 = -1/200;
a13 = 0;
a21 = -1/200;
a22 = 1/200 + 1/250 -1/(j*25) + 1/300;
a23 = -1/300;
a31 = 0;
a32 = -1/300;
a33 = 1/300+1/350-1/(j*16.67);
A = [a11,a12,a13;a21,a22,a23;a31,a32,a33]

   0.0217 + 0.0200i  -0.0050                  0
  -0.0050             0.0123 + 0.0400i  -0.0033
        0            -0.0033             0.0062 + 0.0600i


B = [1/100;0;0]

    0.0100
         0
         0

V = inv(A)*B*X

  24.2853 -23.2416i
  -1.7971 - 3.5726i
  -0.2066 + 0.0785i
```

meaning

$$V_1(t)=24.28\cos(2t)+23.24\sin(2t)$$

$$V_2(t)=-1.79\cos(2t)+3.57\sin(2t)$$

$$V_1(t)=0.21\cos(2t)+0.08\sin(2t)$$

If you prefer polar representation:

```
>>abs(V)

   33.6147
    3.9991
    0.2210


>> angle(V)*180/pi

  -43.7420
 -116.7040
  159.1878
```

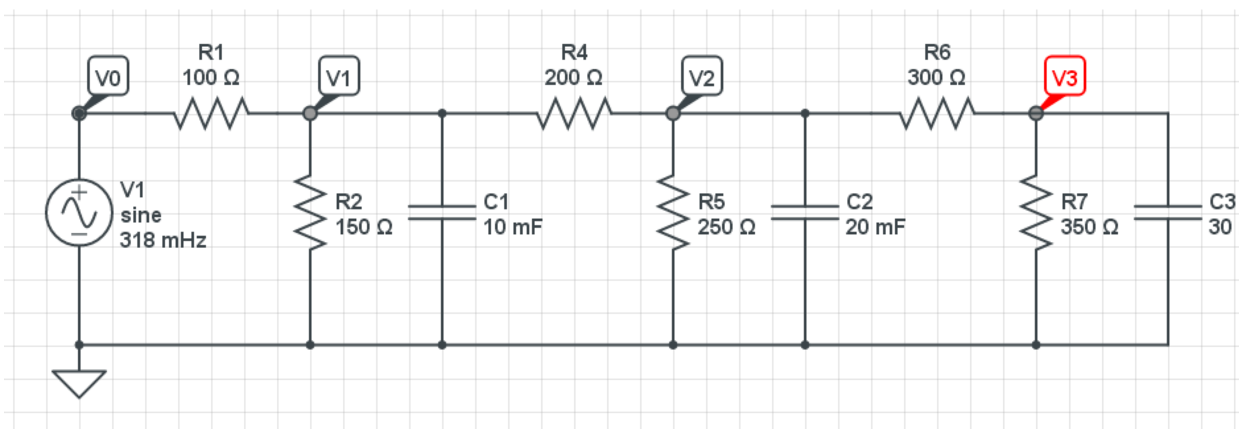
meaning

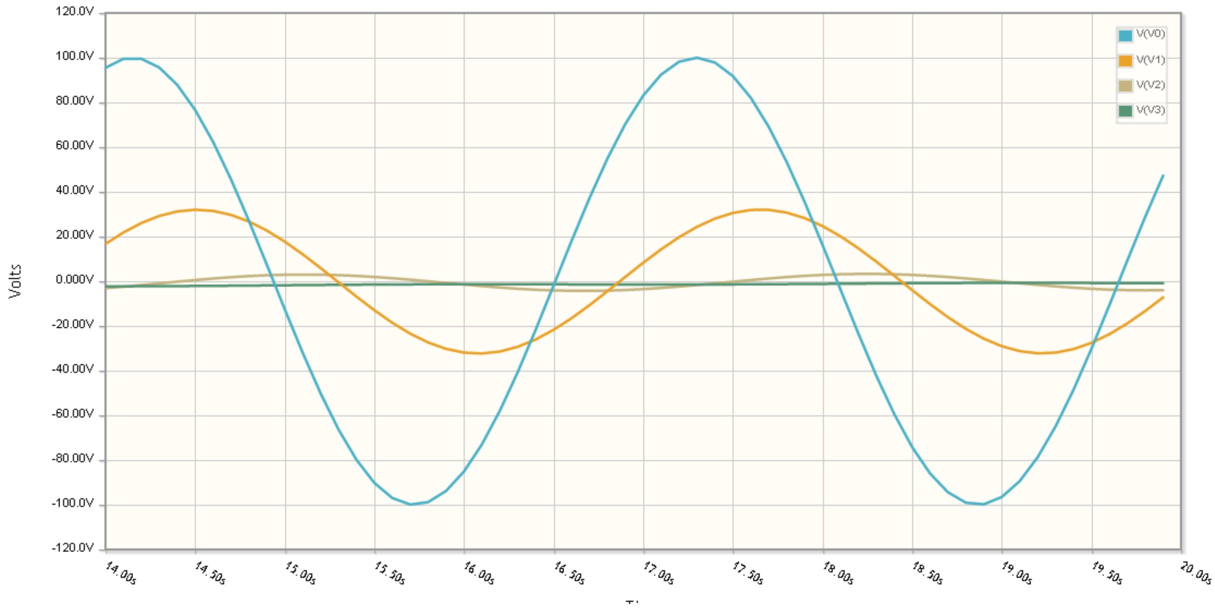
$$V_1(t) = 33.61\cos(2t - 43.7^0)$$

$$V_2(t) = 3.999\cos(2t - 116.7^0)$$

$$V_3(t) = 0.22\cos(2t + 159.2^0)$$

CircuitLab Simulation





Note that V1 has

- A peak of 33.686V (vs. 33.61V computed)
- A delay of

$$\left(\frac{0.4\text{ div}}{3.2\text{ div}}\right)360^0 = 45^0 \quad \text{(vs. 43.7 degrees computed)}$$

Likewise, V2 and V3 match our calculations

	Vin	V1	V2	V3
Calculated	100V	33.61V	3.999V	220mV
CircuitLab	100V	33.678V	4.233V	592 mV