Complex Numbers

Objective:

- Become familiar with using complex numbers for addition, subtraction, multiplication, and dificion

Complex Numbers:

One of the greatest inventions in mathematics was the concept of the number zero. With the number zero, you can express the absence of something. It allows us to use a decimal number system rather than roman numerals. For example, try adding the following numbers using Roman numerals:

MXXIII + CVI =

If this is hard, try adding the following numbers:

1023 + 106 = 1129

Zero allows you to use decimal numbers where it is possible to have numbers like 106 (zero tens).

Another great invention was negative numbers. This allows you to treat profit and loss alike mathematically - the only difference is the sign. Before negative numbers were invented, accountants had to represent loss as a positive debit, profit as a positive credit, and try to keep track of which column each entry belonged. The doule-enty bookkeeping system was significant enough that a small country, like Holland, was able to compete against large contries, like France, simply because they were better at keeping track of what ventures were profitable and which were not. It also resulted in England asking Holland to take over their govenrmnent (hence the German kings of England.)

Complex numbers allow you to respresent sines and cosines with a single number. This allows you to solve a differential equation with sinusoidal inputs by solving a single equation rather than two equations. The catch is the numbers you're using are complex.

Two basic definitions for complex numbers are

$$j^2 = -1$$

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

Polar and Rectangular Form:

A complex number can be represented in rectangular form:

$$x + jy$$

or polar form

$$r \cdot e^{j\theta} = r\angle\theta$$

The relationship is

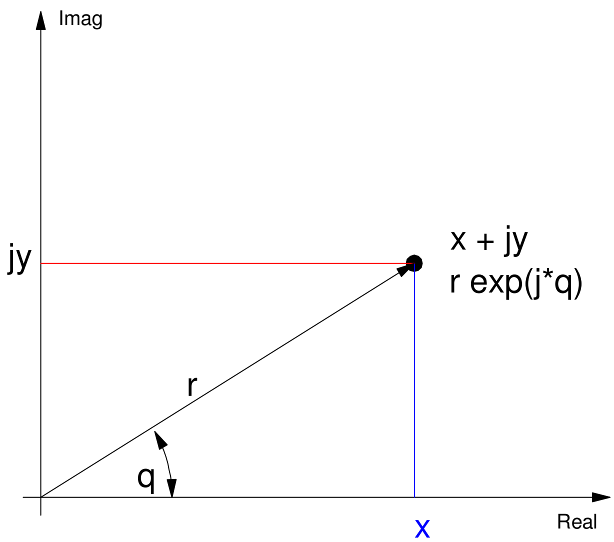
$$r = \sqrt{x^2 + y^2}$$

$$\tan(\theta) = \frac{y}{x}$$

or

$$x = r \cdot \cos(\theta)$$

$$y = r \cdot \sin(\theta)$$



Complex Algebra:

A typical problem we'll be trying to solve is as follows:

Evaluate:

$$\left(\frac{2s+3}{s^2+5s+6}\right)_{s=-2+j3}$$

To evaluate this, we need to know how to add, subtract, multiply, and divide complex numers:

Addition: Just add real plus real, complex plus complex:

$$(2+j3)+(4+j5) = (2+4)+j(3+5) = 6+j8$$

Subtraction: Ditto

$$(2+j3)-(4+j5)=(2-4)+j(3-5)=-2-j2$$

Multiplcation: Multiply and include all cross terms:

$$(2+j3)(4+j5)=(2\cdot 4)+(2\cdot j5)+(j3\cdot 4)+(j3\cdot j5)$$

$$=(8)+(j10)+(j12)+(j^2 15)$$

Note that j2 = -1:

$$=(8-15)+j(10+12)$$

$$=-7+j22$$

Multiplication is easier in polar form:

$$(a\angle\theta)(b\angle\phi)=ab\angle\theta+\phi$$

or

$$(a\cdot e^{j\theta})(b\cdot e^{j\phi})=ab\cdot e^{j(\theta+\phi)}$$

Complex Conjugates:

A useful property of complex numbers is the complex conjugate (denoted as *):

$$(x+jy)^*=x-jy$$

A number multiplied by its complex conjugate is

$$(x+jy)(x-jy)=(x^2+jxy-jxy-j^2y)=x^2+y^2$$

The result is the sum of the square of the real and complex parts. This is also the magnitude squared.

Division: In polar form:

$$(a\angle\theta)(b\angle\phi)^{-1}=\frac{a}{b}\angle\theta-\phi$$

or

$$(a\cdot e^{j\theta})(b\cdot e^{j\phi})^{-1}=\frac{a}{b}\cdot e^{j(\theta-\phi)}$$

Division in rectangular form:

$$\frac{2+j3}{4+j5}=?$$

Multiply by the complex conjugate of the denominator:

$$\left(\frac{2+j3}{4+j5}\right)\left(\frac{4-j5}{4-j5}\right)=\left(\frac{2\cdot 4-j2\cdot 5+j3\cdot 4-j^2 3\cdot 5}{4\cdot 4-j4\cdot 5+j4\cdot 5-j^2 5\cdot 5}\right)=\left(\frac{23+j2}{41}\right)=0.561+j0.0488$$

Note: You'll be doing a lot of complex algebra in ECE. I'd recommend a calculator such as the HP35s - it does complex angebra with ease.

Strange complex numbers:

Find the solution to

$$(1+j2)^{3+j4}$$

Solution: express in exponential (polar) form:

$$1+j2 = \sqrt{5}\,\angle 1.1071 = e^{0.8047+j1.1071}$$

Raise this to the (3+j4) power: (multiply the exponent by (3+j4))

$$\left(e^{0.8047+j1.1071}\right)^{3+j4} = e^{-2.0143+j6.5401}$$

So the answer is

$$e^{-2.0143+j6.5401} = e^{-2.0143}e^{j6.5401}$$

$$(1+j2)^{3+j4} = 0.1334\angle 6.54 = 0.1290 + j0.0339$$