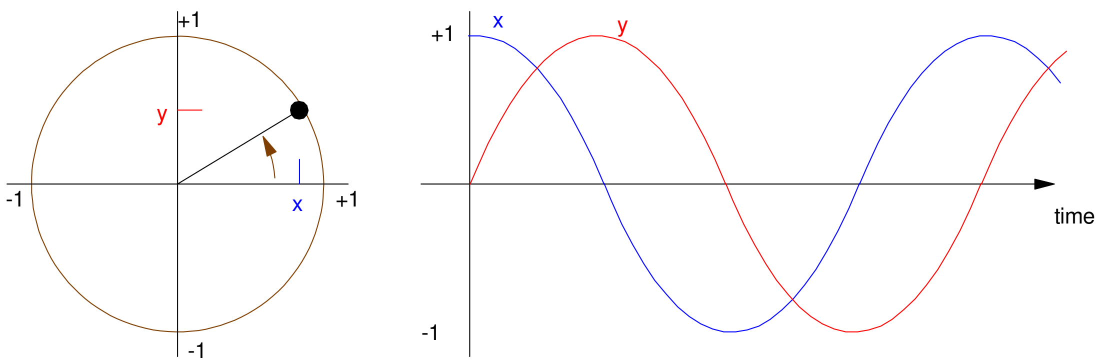
Sinusoidal Sources

So far, we have looked at constant sources. If you want to look at time-varying signals, such as music, you need to expand our analysis to sineusoidal sources.

What Is a Sine Wave?

Circles and Sine Waves: If you take a wheel and spin it counter clockwise,

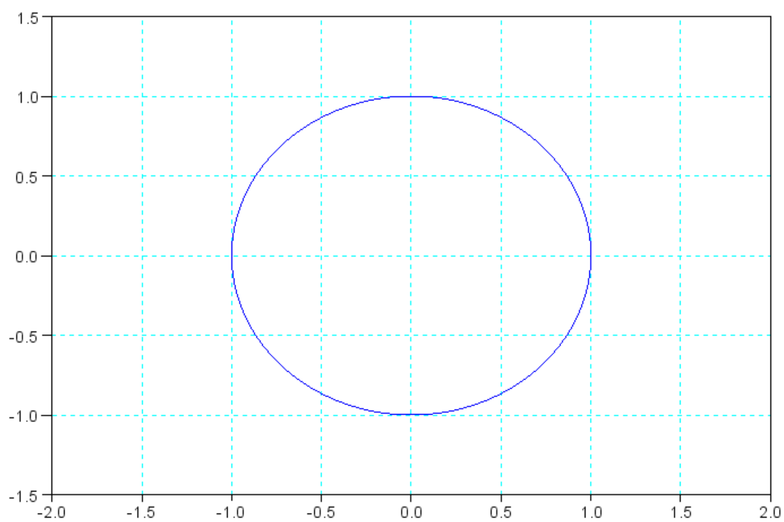
- The x-position of a point on the wheel maps out a cosine function, and
- The y-poisition of a point maps out a sine function



plotting x vs. y where x = cos(q) y = sin(q) produces the unit circle

If you plot cos(q) vs. sin(q), you get the unit circle

```
q = [0:0.001:1]' * 2 * pi;
x = cos(q);
y = sin(q);
plot(cos(q), sin(q))
```



plotting cos(q) vs. sin(q) produces the unit circle

In polar corrdinates, if you plot

$$r = \cos q$$

or

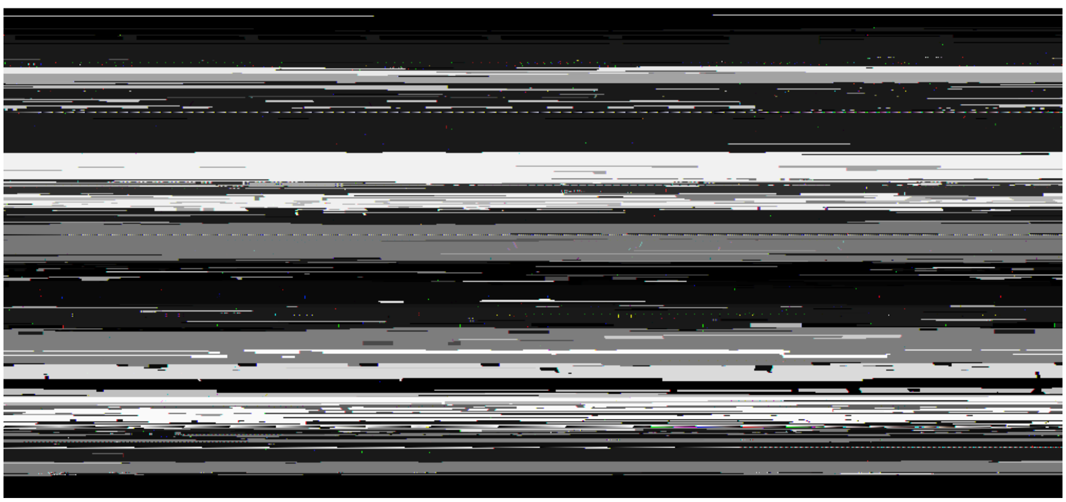
$$r = \sin q$$

you also get circles

```
q = [0:0.001:1]' * 2 * pi;

r1 = cos(q);
x1 = r1.*cos(q);
y1 = r1.*sin(q);

r2 = sin(q);
x2 = r2.*cos(q);
y2 = r2.*sin(q);
plot(x1,y1,'b',x2,y2,'r');
```



r = cos(q) (blue) and r = sin(q) (red) also produce circles

Natural Response to Differential Equations: The natural response to a linear 2nd-order differential equation

$$\frac{d^2y}{dt^2} + w^2 y = 0$$

is a sine wave:

$$y(t) = a\cos(wt) + b\sin(wt)$$

Example: A mass-spring system satisfies the differential equation:

$$F = m\frac{d^2y}{dt^2} = -ky$$

Find the response, y(t) assuming

- m = 1kg
- k = 9 N/m
- y(0) = 2m
- y'(0) = 1m/s

Solution: The differential equation is

$$\frac{d^2y}{dt^2} = -9y$$

Using methods from Calculus, 'guess' the answer of the form

$$y(t) = a\cos(wt) + b\sin(wt)$$

Take the 1st derivative:

$$\frac{dy}{dt} = -aw\sin(wt) + bw\cos(wt)$$

Take the 2nd derivative

$$\frac{d^2y}{dt^2} = -aw^2\cos(wt) - bw^2\sin(wt)$$

$$= -w^2(a\cos(wt) - b\sin(wt))$$

$$= -w^2y$$

Substitute

$$\frac{d^2y}{dt^2} = -9y$$

$$-w^2y = -9y$$

$$w = 3$$

and

$$y(t) = a\cos(3t) + b\sin(3t)$$

Plugging in the initial conditions:

$$y(0) = 2 = a$$

$$y'(0) = 1 = -3b$$

This results in

$$\boxed{y(t) = 2\cos(3t) - 0.333\sin(3t)}$$

Why Use Sine Waves?

There are two main reasons we use sine waves (instead of square waves or other time-varying signals)

- Sine waves is sine waves are eigenfunctions: if you have a differential equation whose input is a sine wave, the output is a sine wave. Not many functions have that property.
- You can decompose any periodic signal into a sum of sine waves (Fourier transform - coming soon). If you can solve a circuit for a sinusoidal input, you can solve it for any periodic input.

Eigenfunctions: Sine waves are the only functions where the solution to a differential equation is the same form as that function. For example, take the differential equation

$$y'' + 3y' + 2y = 2x$$

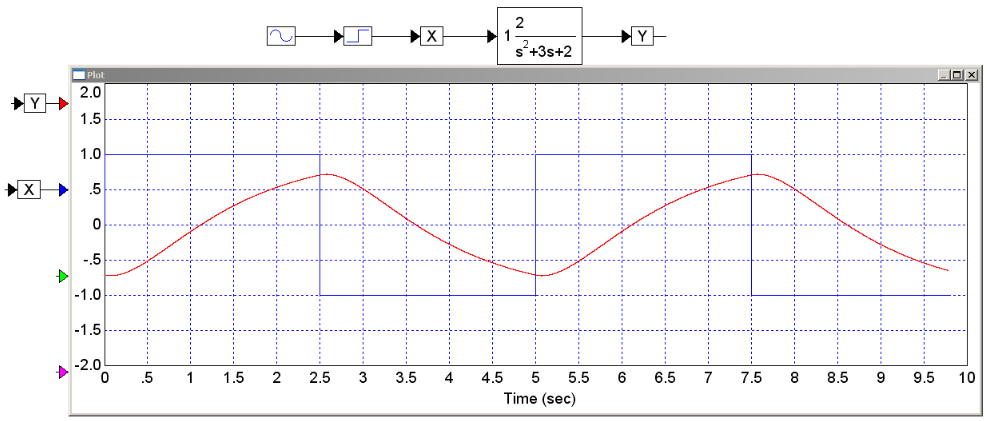
In transfer function form, this can be written as

$$(s^2 + 3s + 2)Y = 2X$$

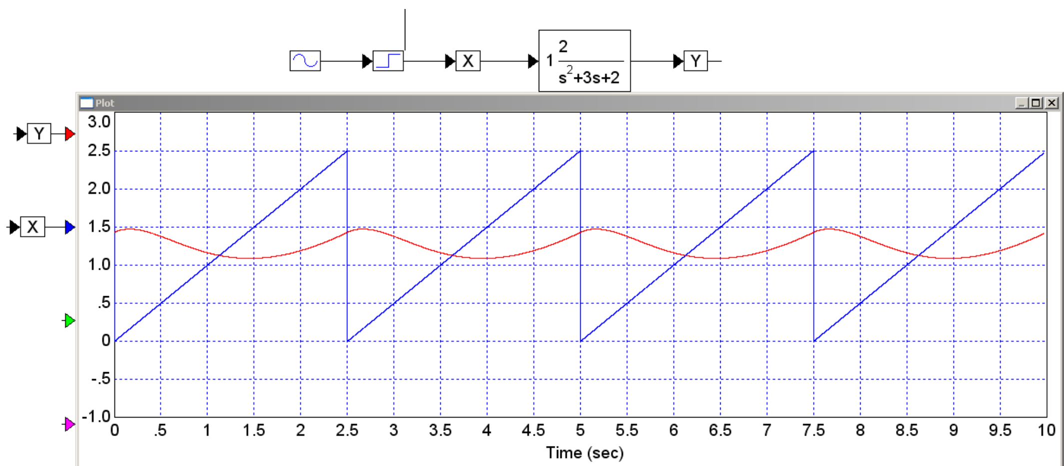
$$Y = \left(\frac{2}{s^2+3s+2}\right)X$$

Later, we'll see that this describes a 2-stage RC filter (as well as other systems). If the input, X, is a pure sine wave, the output is a pure sine wave at the same frequency:

No other periodic function has this property: square waves don't do this:



Sawtooth waves don't do this:



Only sine waves have the property that the output is the same shape as the input, only with a change in amplitude and with a time delay.

Fourier Transform: (Covered in ECE 311 Circuits II)

A second reson for using sine waves is that they are versitile: you can represent any periodic signal as a sum of sine waves.

For example, a half-rectified sine wave

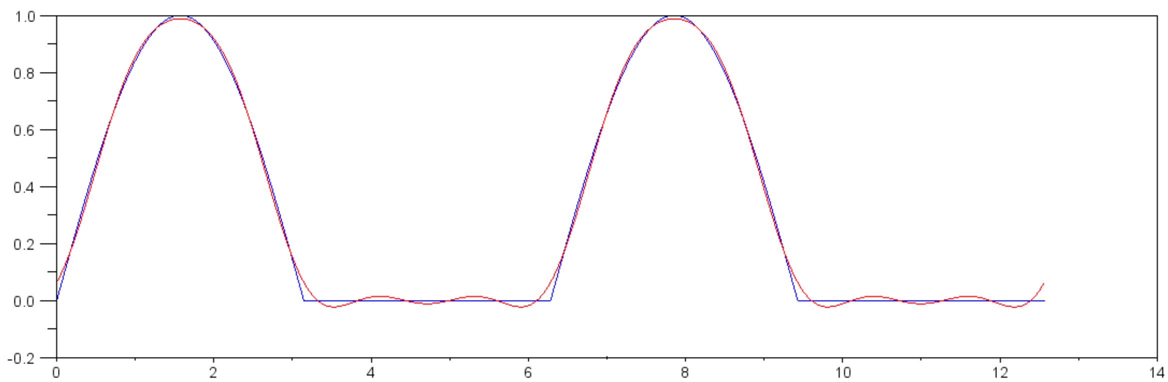
$$x(t) = \begin{cases} \sin(t) & \sin(t) > 0 \\ 0 & otherwise \end{cases}$$

can be expressed as the series

$$x(t) = \frac{1}{\pi} + \frac{1}{2}\sin(t) + \sum_{n \text{ even}} \left(\frac{2}{\pi\left(n^2-1\right)} \right) \cos(nt)$$

```
t = [0:0.001:2]' * 2*pi;
x = max(0, sin(t));

y = 1/pi + 0.5*sin(t) - 2/(3*pi)*cos(2*t) - 2/(15*pi)*cos(4*t);
plot(t,x,'b',t,y,'r')
```



Sine Wave Definitions

To alleviate some of the confusion, some definitions are needed.

Vp: Peak Voltage: The amplitude of the sine wave from it's average voltage (usually zero).

Vpp: Peak to Peak Voltage: The distance between the maximum and minimum voltage. Vpp = 2 Vp

Vrms: rms Voltage: The DC votlage which would produce the same amount of heat through a 1 Ohm resistor.

Period (seconds): Time time between zero crossings (or peak votlages)

Frequency (Hz): One over the period

Frequency (rad/sec): The natural frequency: 1Hz = 2π rad/sec.

You can convert from one to the other

$$V_p = \frac{1}{2} V_{pp}$$

$$V_{rms} = \frac{1}{\sqrt{2}} V_p$$

Derivation of rms voltage: Assume V is 1 rad/sec a sine wave

$$v(t) = V_p \sin(t)$$

The instantaneous power through a 1 Ohm resistor is

$$P(t) = \frac{v^2}{R} = (V_p \sin(t))^2$$

The average power through a 1 Ohm resistor is

$$P_{avg} = \frac{1}{2\pi} \int_0^{2\pi} (V_p \sin(t))^2 dt$$

$$P_{avg} = \frac{1}{2\pi} \cdot \int_0^{2\pi} \left(V_p^2 \cdot \left(\frac{1-\cos(2t)}{2} \right) \right) dt$$

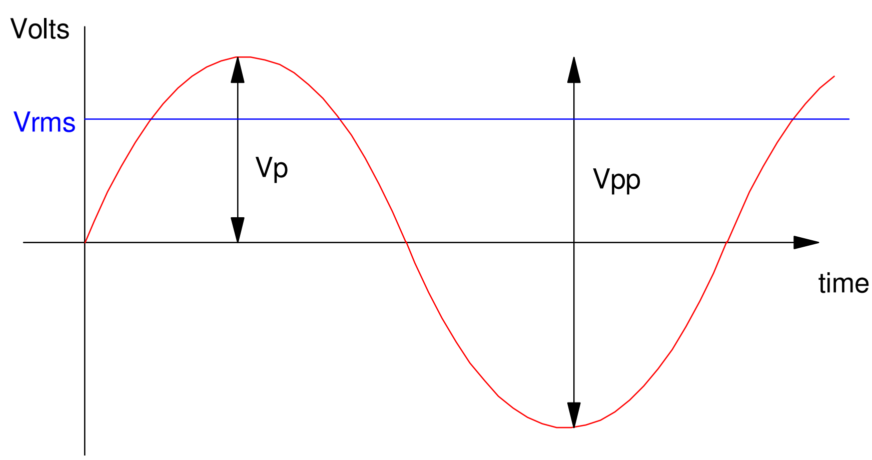
$$P_{avg} = \frac{V_p^2}{2}$$

The DC voltage which produces the same power is the rms voltage

$$\left(\frac{V_{rms}^2}{R} \right) = V_{rms}^2 = \frac{1}{2} V_p^2$$

$$\boxed{V_{rms} = \frac{1}{\sqrt{2}} V_p}$$

Vp, Vpp, and Vrms are different. Likewise, when dealing with sinusoids, you have to specify what units you're talking about.



Relationship between Vp, Vpp, and Vrms

The same holds for current:

I_p: Peak Current The maximum current relative to the average current

I_{pp}: Peak-to-Peak Current: The maximum current minus the minimum current

I_{rms}: rms Current: The DC current which produces the same amount of heat through a 1 Ohm resistor.