EE 311 Circuits II: Phasors

ECE 111 Introduction to ECE

Jake Glower - Week #13

Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

ECE 311 Circuits 2: Phasors

Topics

- Phasors
- Representing voltages using phasors
- Representing RLC using phasors
- AC circuit analysis using phasors
- HP42 Calculator

Introduction

DC Analysis: First part of Circuits I

With DC circuits:

- Voltages can be expressed by a real number
- Currents can be expressed with real numbers, and
- Resistance's can be expressed with real numbers.

AC Analysis: End of Circuits I, all of Circuits II

- Voltages have two terms: sine & cosine
- Impedances include Resistors, Inductors, & Capacitors

Complex numbers are needed for AC analysis

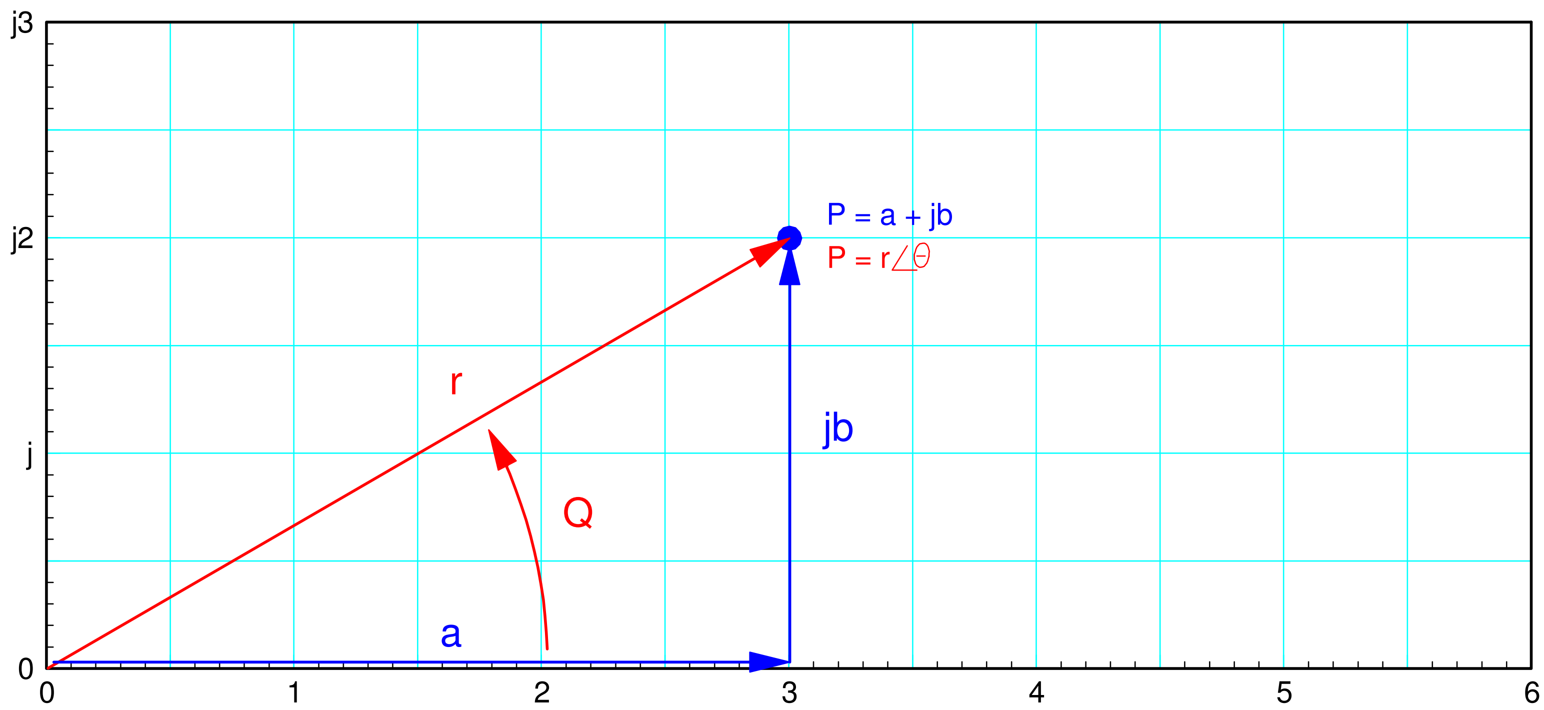
Representation of Complex Numbers:

Rectangular Form:

$$P = a + jb$$

Polar Form

$$P = r\angle\theta$$



Phasor Representation of Voltages

Euler's identity states

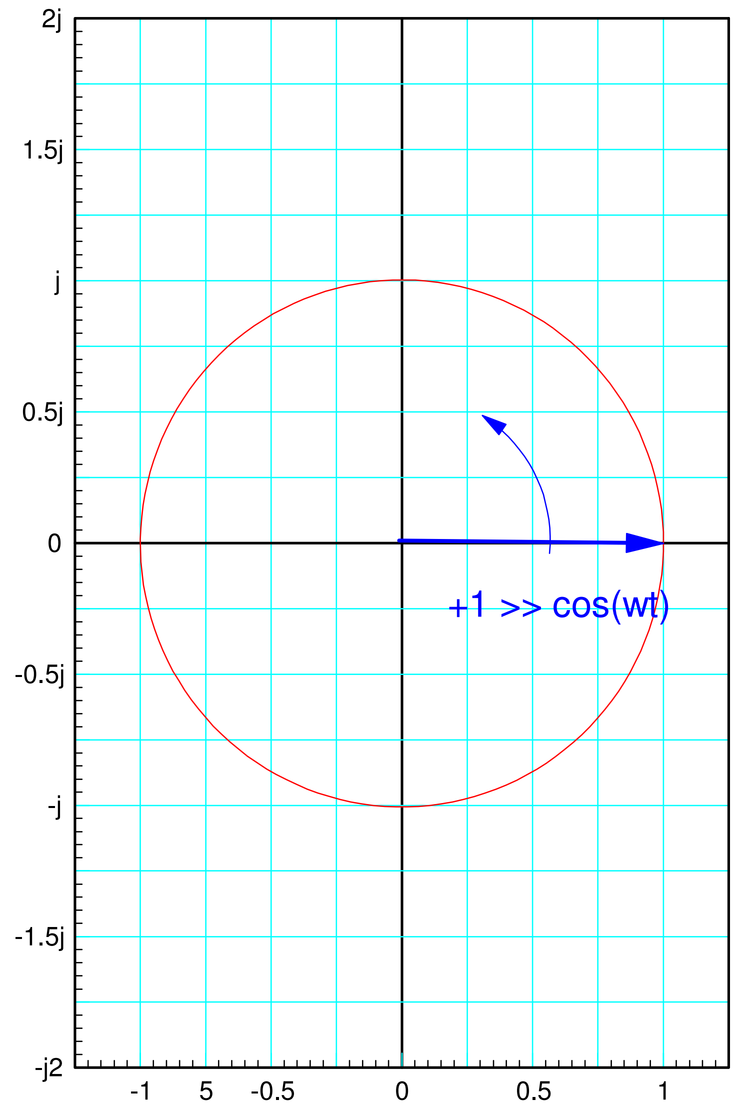
$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

If you assume all functions are of the form of $e^{j\omega t}$, then

$$\cos(\omega t) = real(e^{j\omega t})$$

Likewise, the phasor (complex-number) representation for cosine is one

$$1 \leftrightarrow \cos(\omega t)$$



If you multiply by (a + jb)

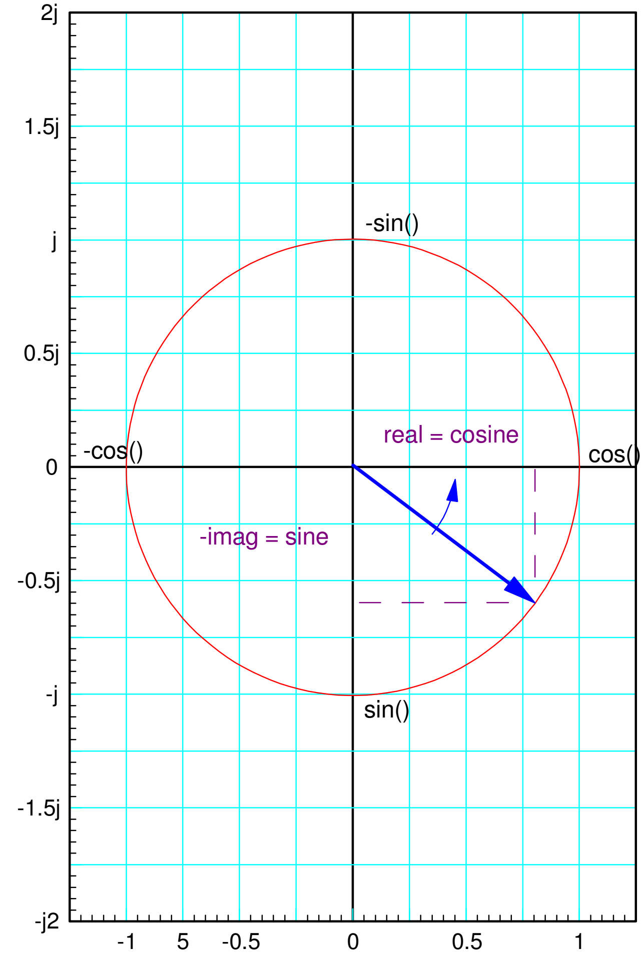
$$(a+jb)e^{j\omega t} = (a+jb)(\cos(\omega t) + j\sin(\omega t))$$
$$= (a\cos(\omega t) - b\sin(\omega t)) + j(\cdots)$$

and take the real part you get the phasor representation for a generalized sine wave

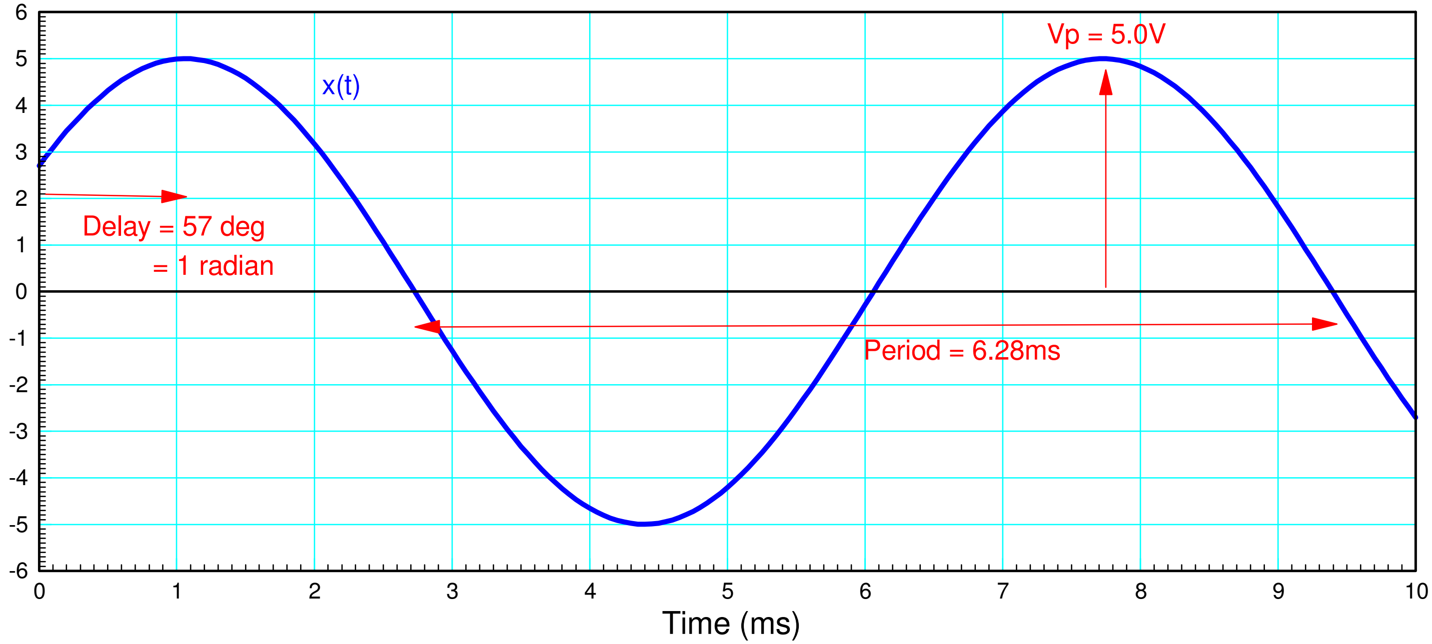
$$a + jb \leftrightarrow a\cos(\omega t) - b\sin(\omega t)$$

You can also represent voltages in polar form:

$$r\angle\theta \leftrightarrow r\cos(\omega t + \theta)$$



Example: Determine x(t) for the following waveform:



Typical Sinusoid: Determine x(t)

The frequency comes from the period

$$T = 6.28ms$$

$$f = \frac{1}{T} = 159.2Hz = 159.2\frac{\text{cycles}}{\text{second}}$$

$$\omega = 2\pi f = 1000\frac{\text{rad}}{\text{sec}}$$

$$\theta = -\left(\frac{\text{1ms delay}}{\text{6.28ms period}}\right)2\pi = -1.0\,\text{radian} = \text{-}57.3^0$$

meaning (in polar form)

$$x(t) = 5\cos\left(1000t - 57.3^0\right) = 5\angle -57.3^0$$

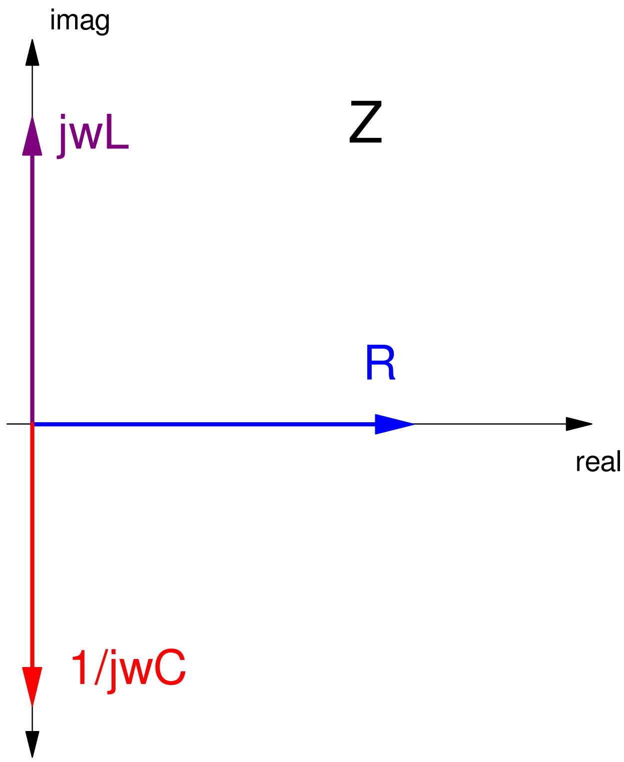
In rectangular form

$$X = 2.70 - j4.21$$

$$x(t) = 2.70\cos(1000t) + 4.21\sin(1000t)$$

Phasor Representation for Impedance's

When dealing with AC signals, resistors, capacitors, and inductors can all be used. Phasor analysis converts each of these to a complex impedance.



Resistors: The VI relationship for a resistor is

$$V = IR$$

The phasor impedance of a resistor is R.

$$R \rightarrow R$$

Inductors: The VI relationship for an inductor is

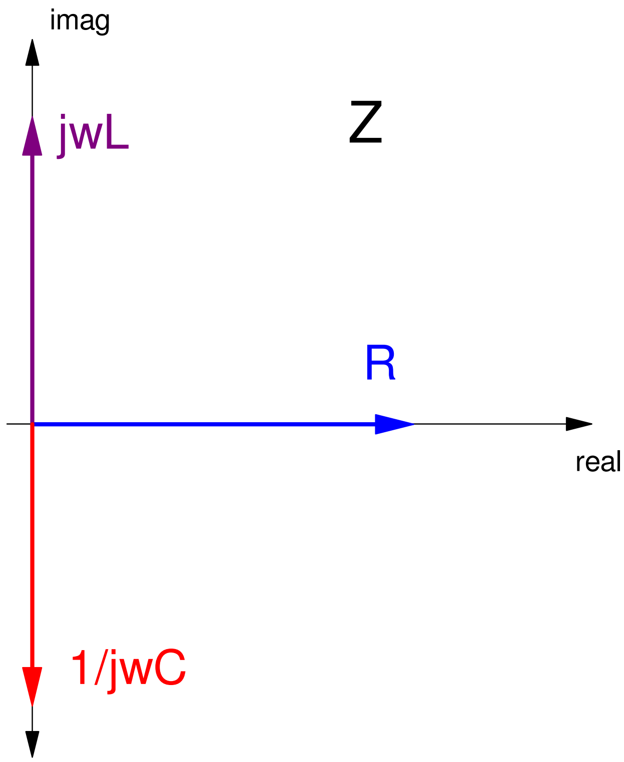
$$V = L\frac{dI}{dt}$$

Assuming all signals are in the form of $e^{j\omega t}$, this means

$$V = L\frac{d}{dt}(e^{j\omega t}) = j\omega L\, e^{j\omega t} = j\omega L \cdot I$$

The impedance of an inductor is $j\omega L$

$$L \rightarrow j\omega L$$



Capacitors: The VI relationship for a capacitor is

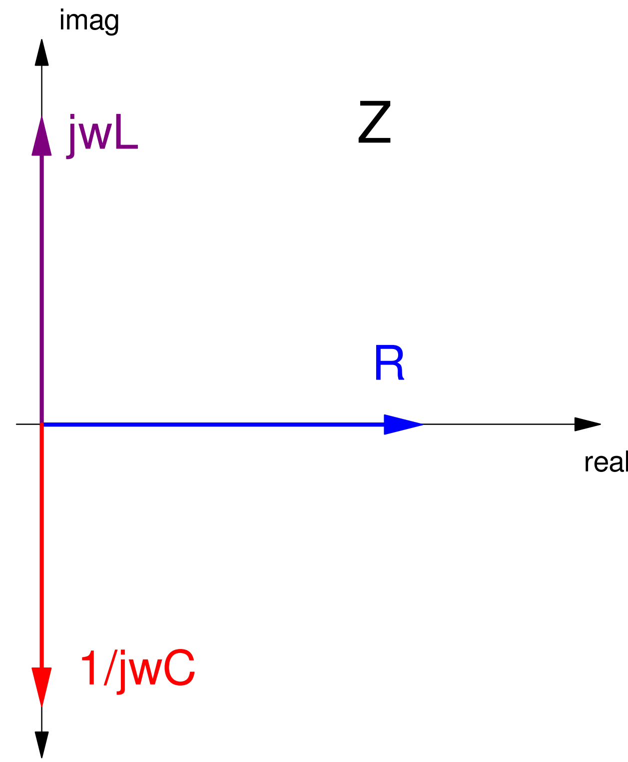
$$V = \frac{1}{C}\int I\,dt$$

Assuming all signals are in the form of $e^{j\omega t}$, this means

$$V = \frac{1}{C}\int (e^{j\omega t});dt = \left(\frac{1}{j\omega C}\right)e^{j\omega t} = \left(\frac{1}{j\omega C}\right)I$$

The impedance of a capacitor is $\left(\frac{1}{j\omega C}\right)$

$$C \rightarrow \frac{1}{j\omega C}$$



ELI the ICE Man

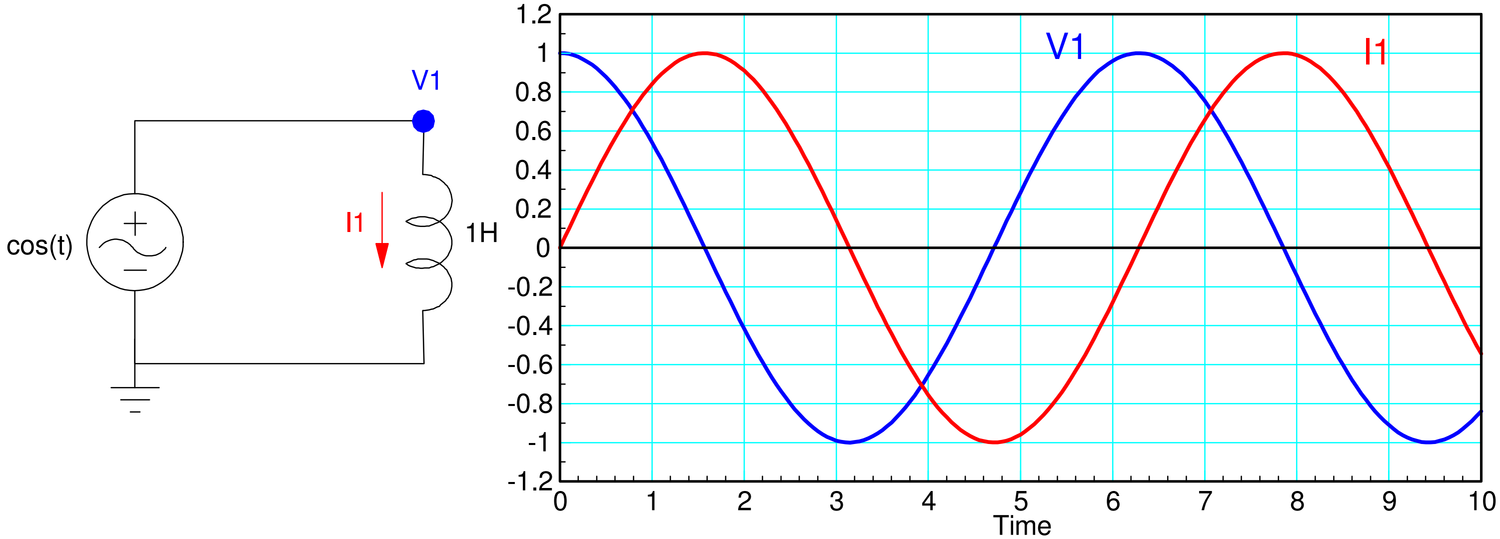
- For inductors (L), voltage leads current (ELI)
- For capacitors (C), current leads voltage (ICE).

ELI: If $\omega L = 1$ then

$$L \rightarrow j\omega L = j = 1\angle 90^0$$

$$V = I \cdot j\omega L = I \cdot 1\angle 90^0$$

Voltage leads current (ELI) for inductors

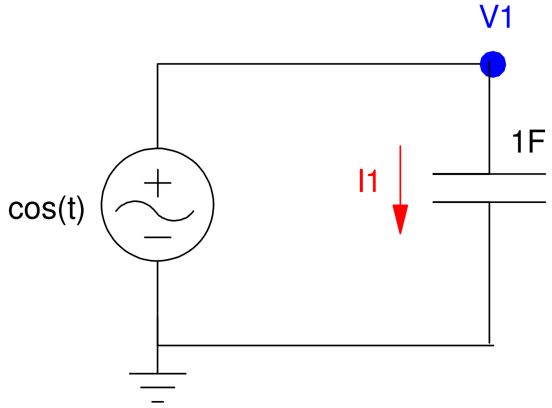


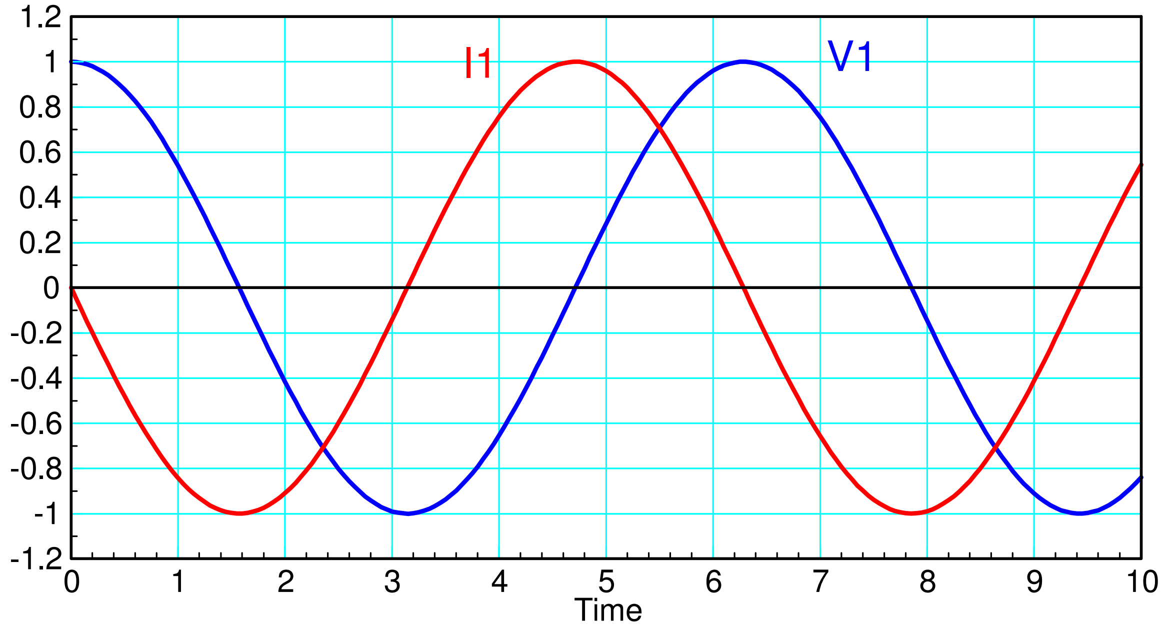
ICE: For capacitors, if $\omega C = 1$, then

$$\frac{1}{j\omega C} = -j = 1\angle -90^0$$

$$V = I \cdot 1\angle -90^0$$

Voltage lags current by 90 degrees for capacitors (ICE)





Phasor Summary

Component	Phasor Representation
V = a cos(wt) - b sin(wt)	a + jb
R	R
L	jwL
C	1 / jwC

Simplification of RLC circuits

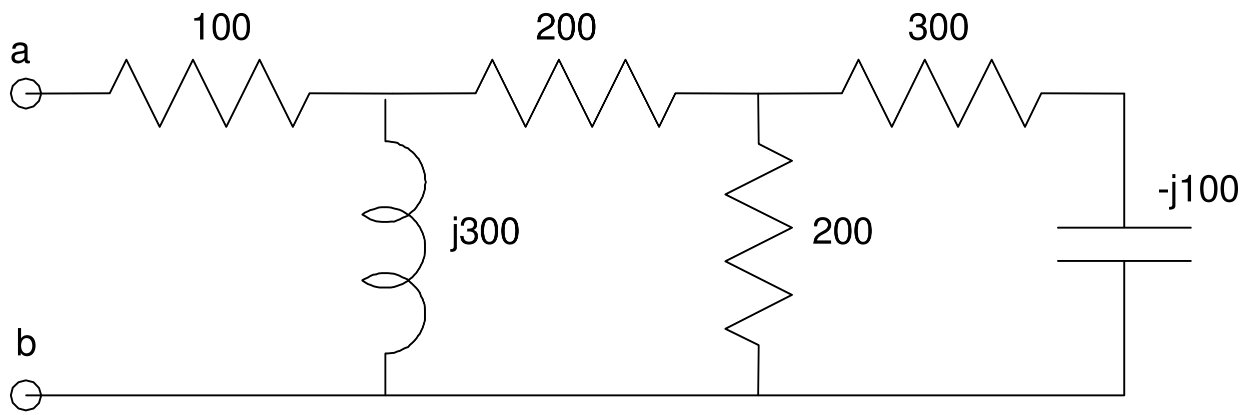
Resistor Circuits:

- Resistors in series add
- Resistors in parallel add as $\left(\frac{1}{R_1}+\frac{1}{R_2}+..\right)^{-1}$

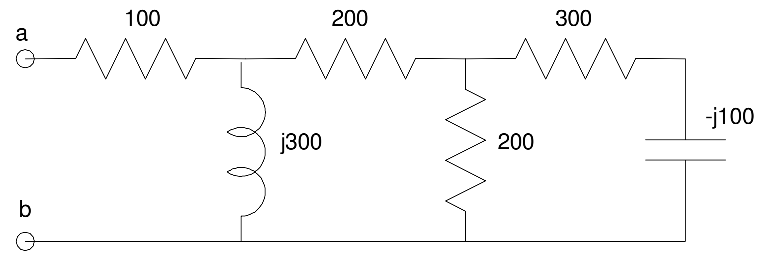
RLC Circuits:

- Same as resistor circuits
- Except you're dealing with complex numbers.

Example: Determine the impedance Zab.



Solution:



-j100 and +300 are in series

$$-j100 + 300 = 300 - j100$$

This is in parallel with 200

$$(200)||(300 - j100) = \left(\frac{1}{200} + \frac{1}{300 - j100}\right)^{-1} = 123.07 - j15.38$$

Which is in series with 200

$$(200) + (123.07 - j15.38) = 323.07 - j15.38$$

Which is in parallel with +j300

$$(j300)||(323.08 - j15.38) = 156.85 + j161.83$$

which is in series with 100

$$(100) + (156.85 + j161.83) = 256.85 + j161.83$$

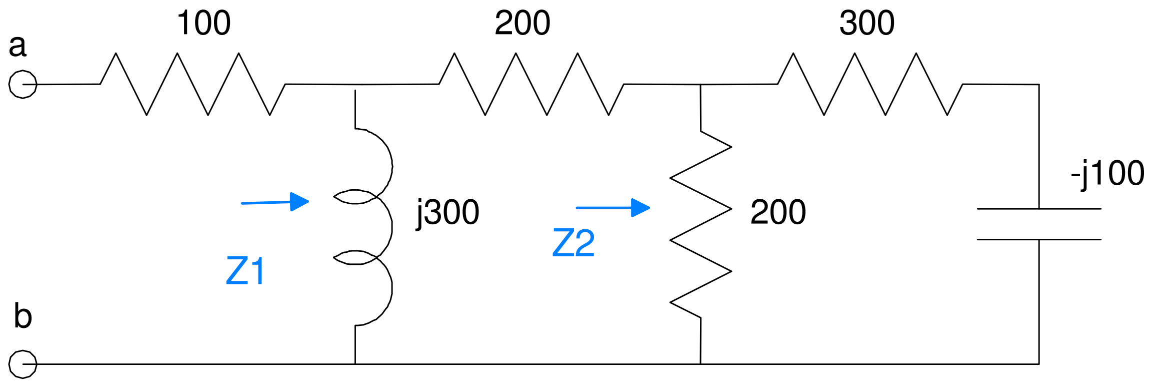
Answer:

$$Z_{ab} = 256.85 + j161.83$$

Solving in Matlab

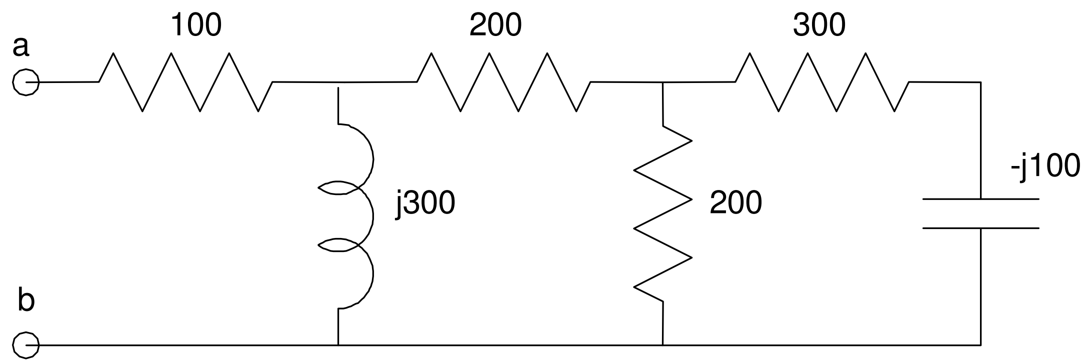
```
>> Z2 = 1 / (1/200 + 1/(300 - j*100) )

Z2 =  1.2308e+002 -1.5385e+001i

>> Z1 = 1 / (1/(j*300) + 1/(200 + Z2) )

Z1 =  1.5685e+002 +1.6183e+002i

>> Zab = 100 + Z1
```

Zab = 2.5685e+002 +1.6183e+002i



Solving with an HP42

```
300
enter
-100
compelx
1/X
200
1/X
+
1/X     Z2 = 123.0769
- j15.3846
200
+
1/X
0
enter
300
complex
1/X
+
1/X     Z1 = 156.8465 + j161.8257
100
+       Zab = 245.8456 + j161.8257
```



Solving in CircuitLab

Zab tells you that current and voltage are related by

$$V = I \cdot Z_{ab}$$

$$V = I \cdot (245.8456 + j161.8257)$$

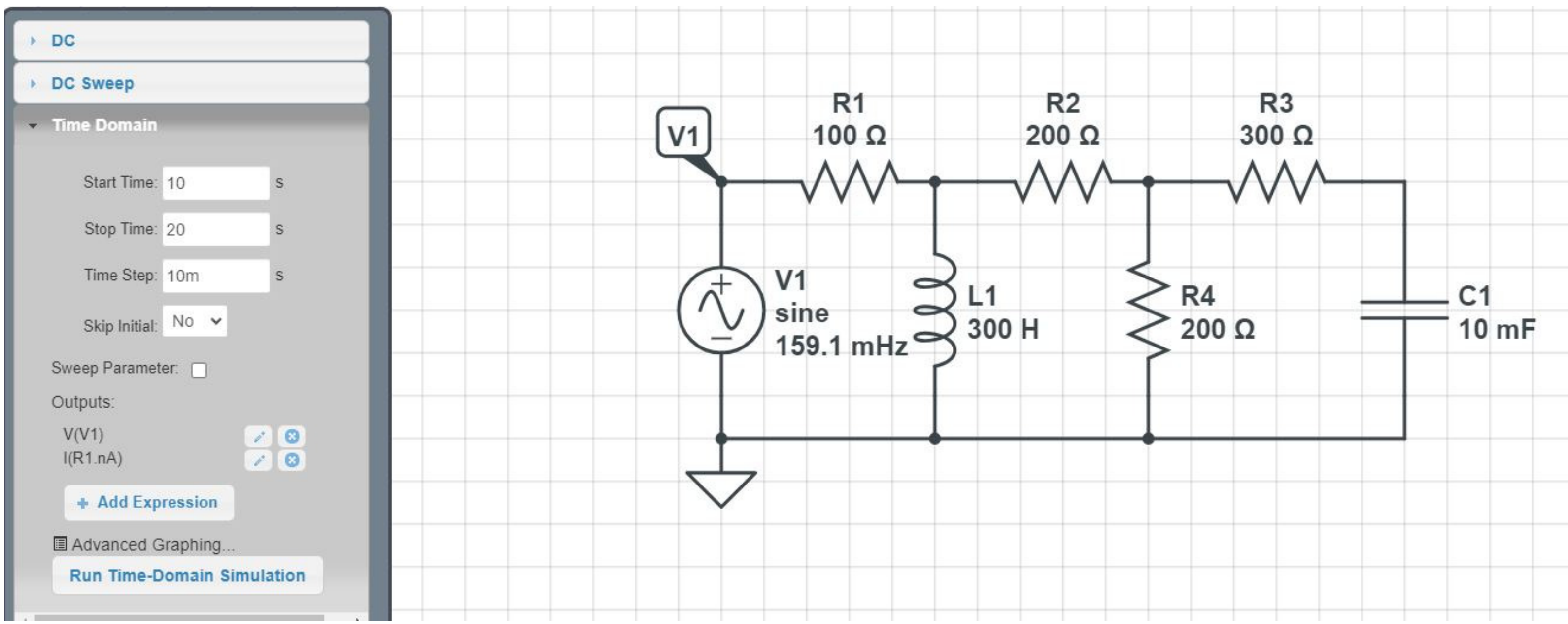
$$V = I \cdot (294.32\angle 33.35^{0})$$

Translation:

- The voltage will be 294.32 times larger than the current
- Voltage leads current by 33.35 degrees

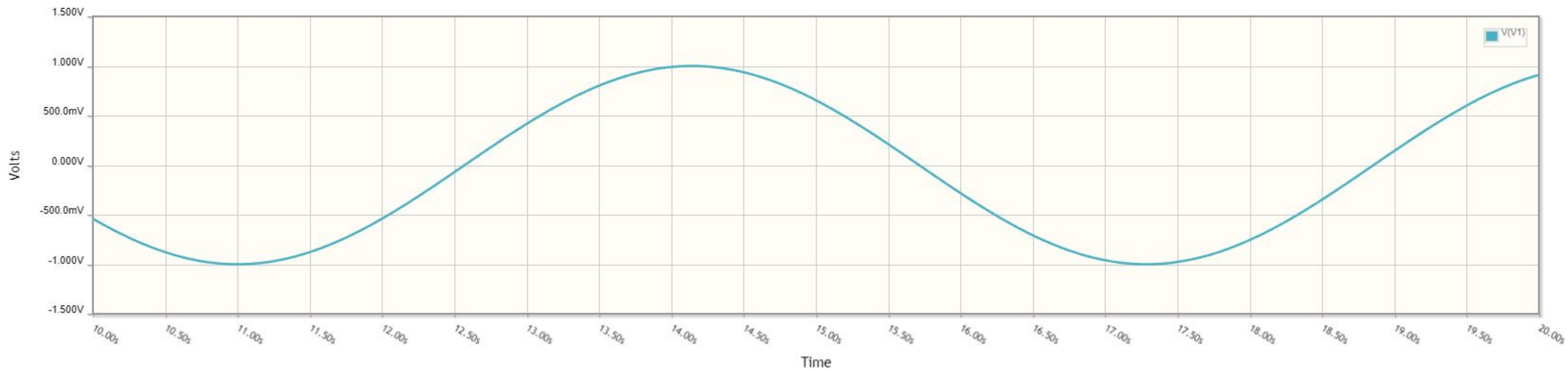
Let $\omega = 1$ rad/sec

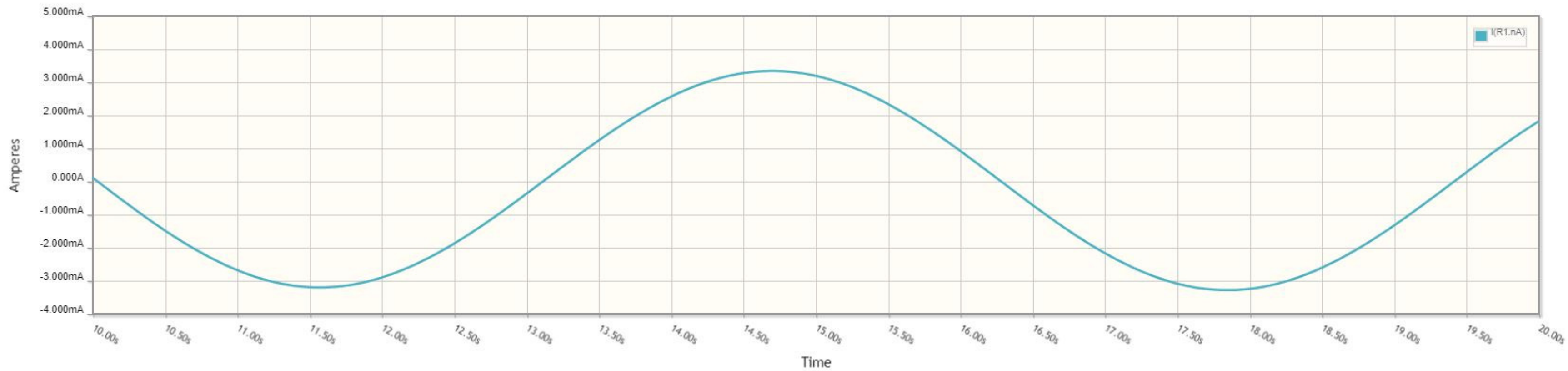
- 0.1591 Hz
- $\omega = 2\pi f$
- $L = Z, \quad C = 1/Z$



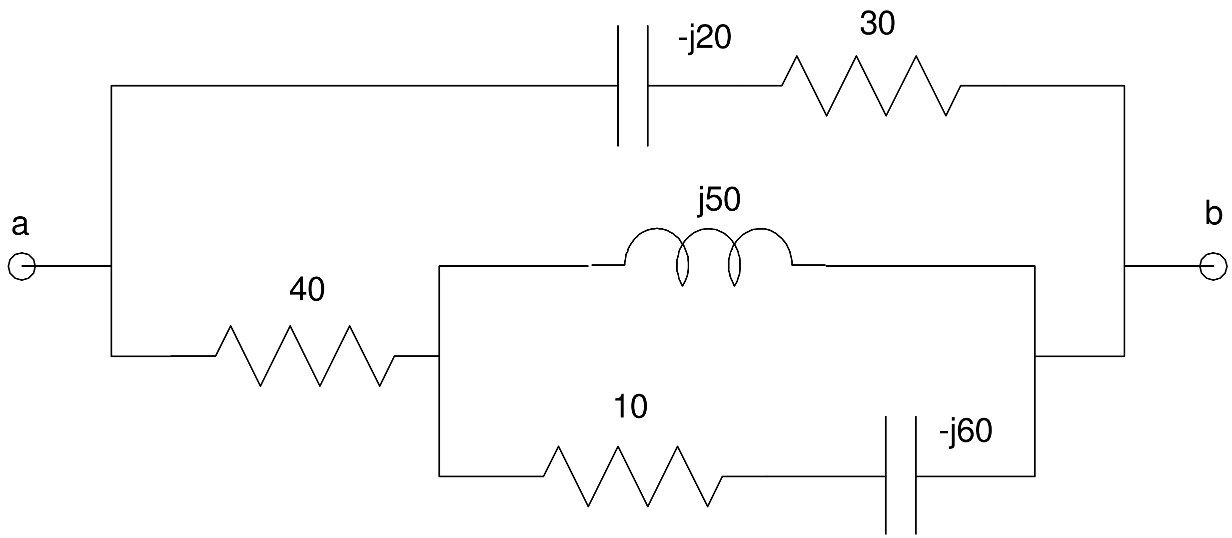
Plot voltage and current

- $Z_{ab} = 294.32\angle 33.35^0$
- The peak voltage is 294 times the peak current
- The peak voltage is 33 degrees before the peak current





Example 2: Determine Zab



Solution

$$(10)+(-j60)=10-j60$$

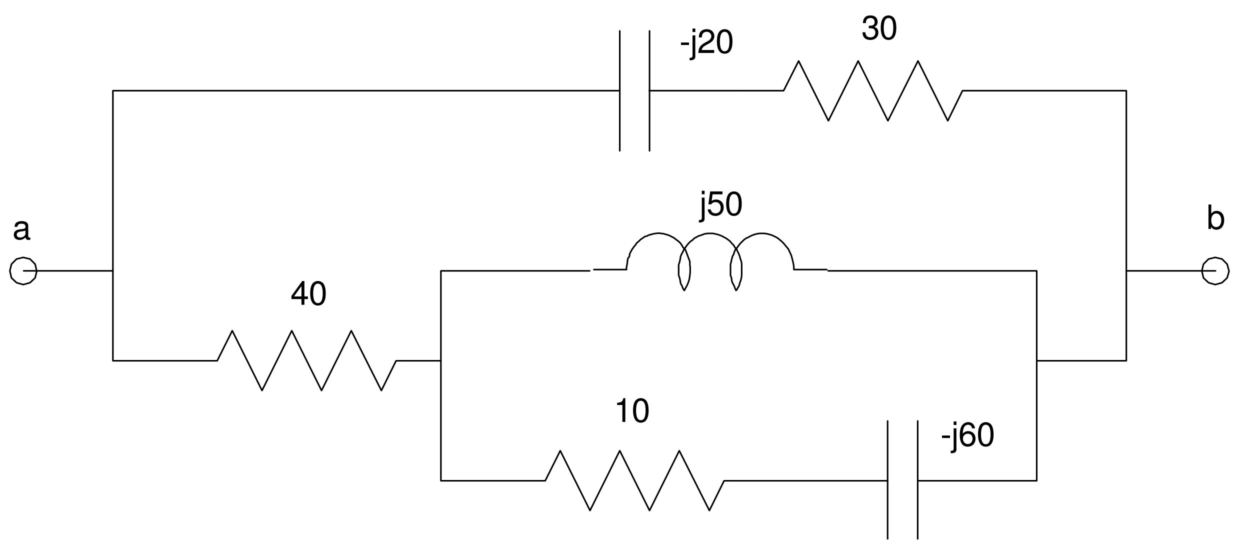
$$(10-j60)||(j50)=125.00+j175.00$$

$$(125.00+j175.00)+(40)=165.00+j175.00$$

$$(165.00+j175.00)||(30-j20)=31.42-j14.98$$

answer:

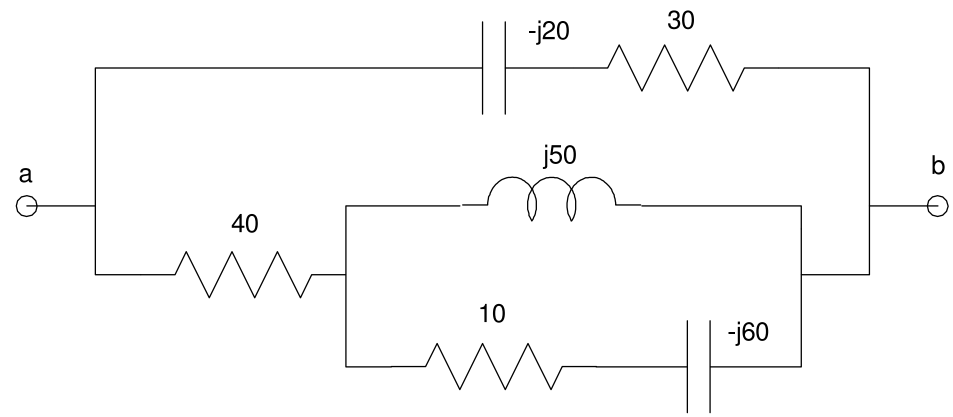
$$Z_{ab}=31.42-j14.98$$



Solve in Matlab:

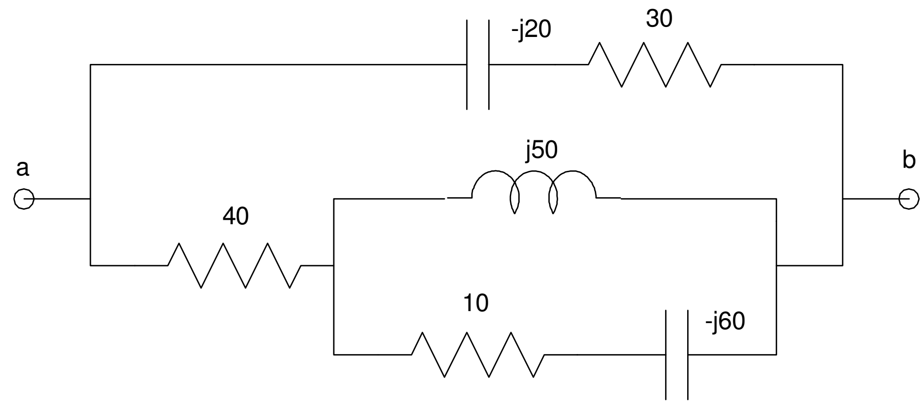
```
>> Z1 = 10 - j*60

Z1 =  10.0000 -60.0000i

>> Z2 = 1 / (1/Z1 + 1/(j*50))

Z2 =  1.2500e+002 +1.7500e+002i

>> Z3 = Z2 + 40

Z3 =  1.6500e+002 +1.7500e+002i

>> Z4 = 1/( 1/Z3 + 1/(30-j*20) )
```

Z4 = 31.4263 -14.9799i



Solve using an HP42

```
10
enter
-60
complex
1/X
0
enter
50
complex
1/X
+
1/X             Z2 = 125 + j175
40
+               Z3 = 165 + j175
1/X
30
enter
-20
complex
1/X
+
1/X             Zab = 31.4263 - j14.9799
```



Solve with CircuitLab:

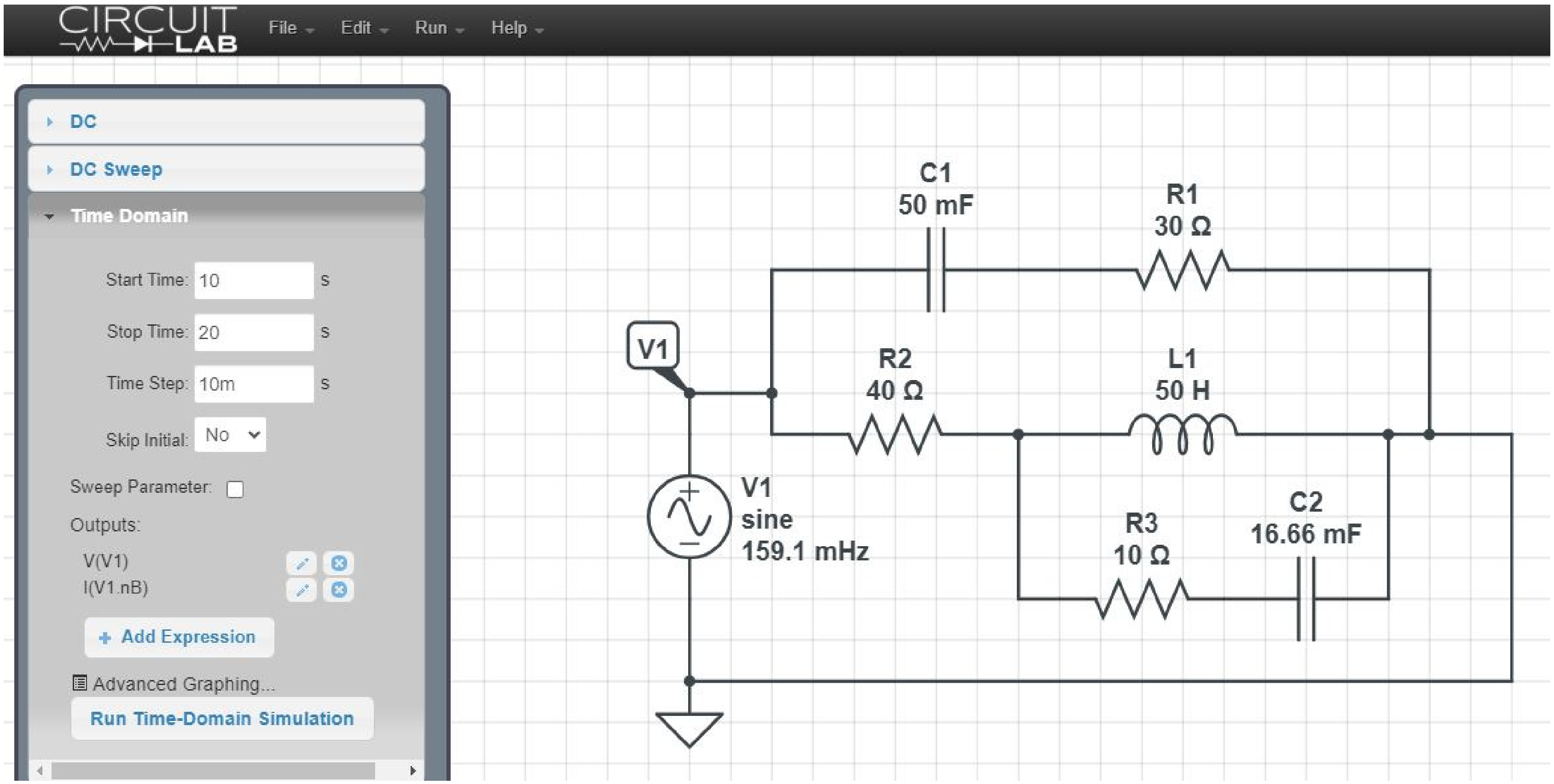
- Same trick as before: Let $\omega = 1$

$$V = I \cdot (31.4263 - j14.9799)$$

$$V = I \cdot (34.81 \angle -25.49^{0})$$

This means

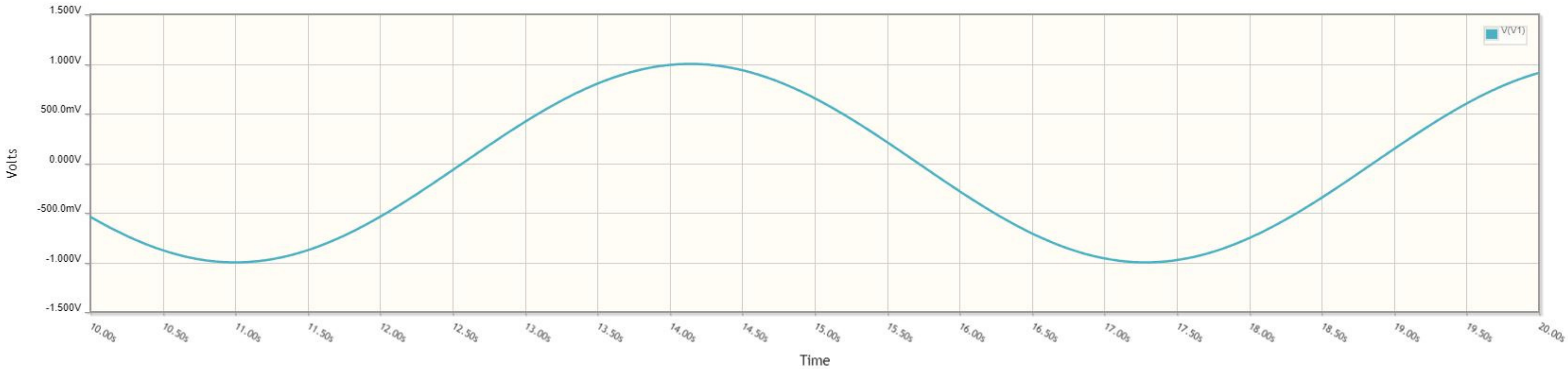
- Voltage should be 34.81 times larger than current
- Voltage should lag behind current by 25.49 degrees.

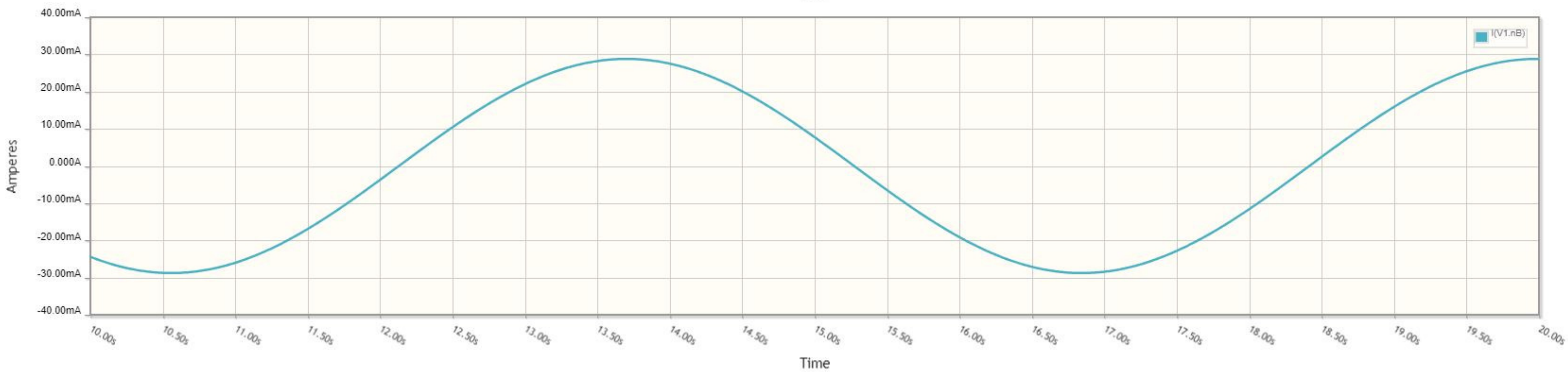


CircuitLab simulation to check the impedance

$Z_{ab} = 34.81\angle -25.49^{0}$

- Voltage is 34.81 times the current
- Voltage lags current by 25.49 degrees



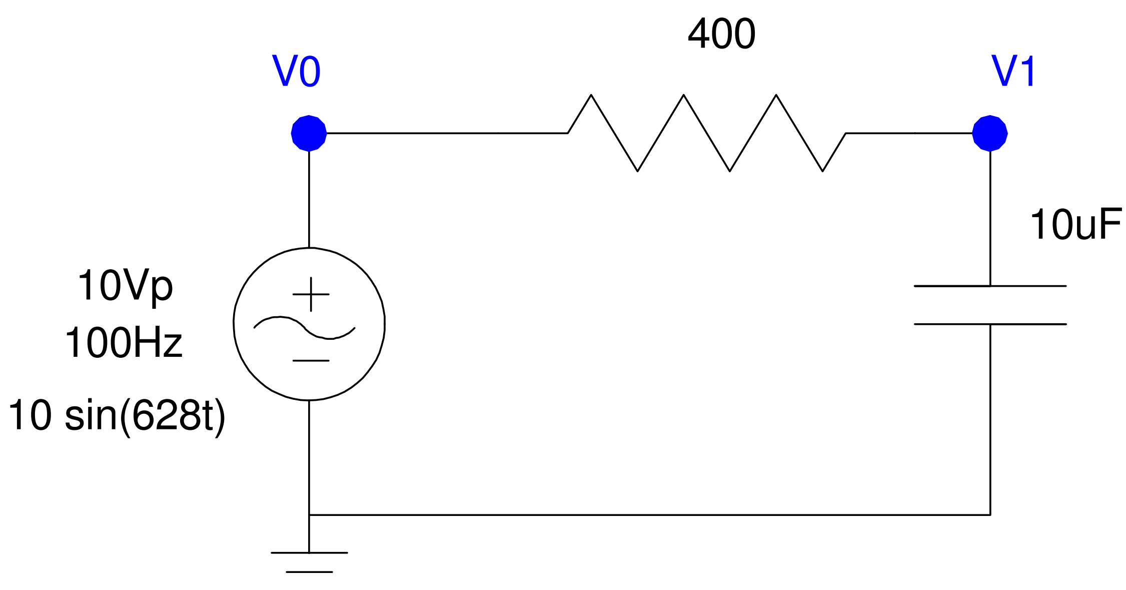


Circuit Analysis with Phasors

Everything we did at DC still works for AC analysis, only now with complex numbers,.

Example 1: RC Circuit

- Determine V1(t)
- $V_0 = 10\sin(628t)$

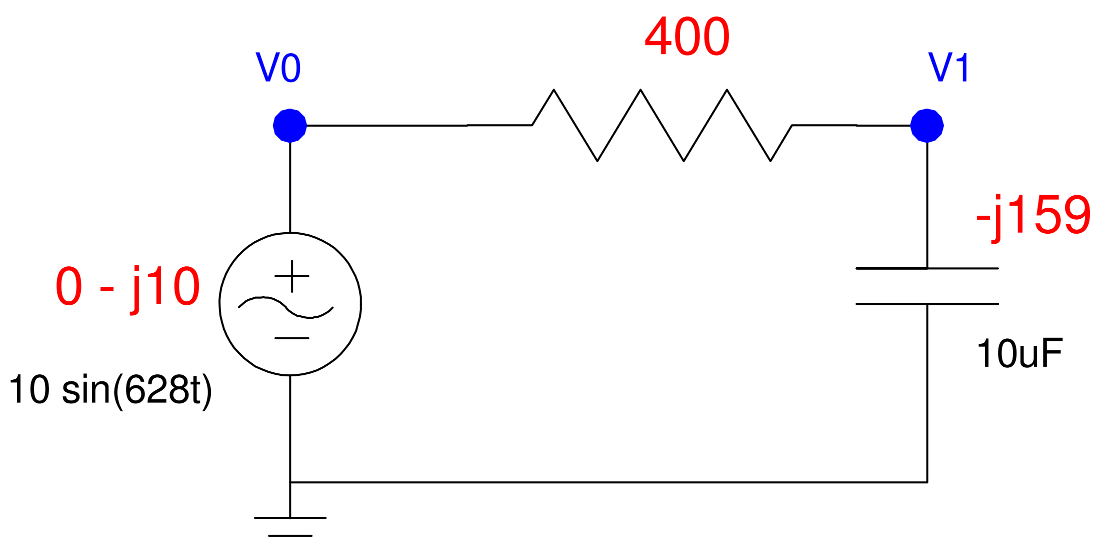


Step 1: Replace the capacitor with its complex impedance. Since the input is 628 rad/sec, that's the frequency you care about

$$\omega = 628 \quad \text{rad/sec}$$

$$Z_c = \frac{1}{j\omega C} = -j159\Omega$$

$$V_0 = 0 - j10$$



Step 2: Solve just like you did with a DC circuit, only with complex numbers

$$V_1 = \left(\frac{-j159}{-j159+400}\right)(0-j10)$$

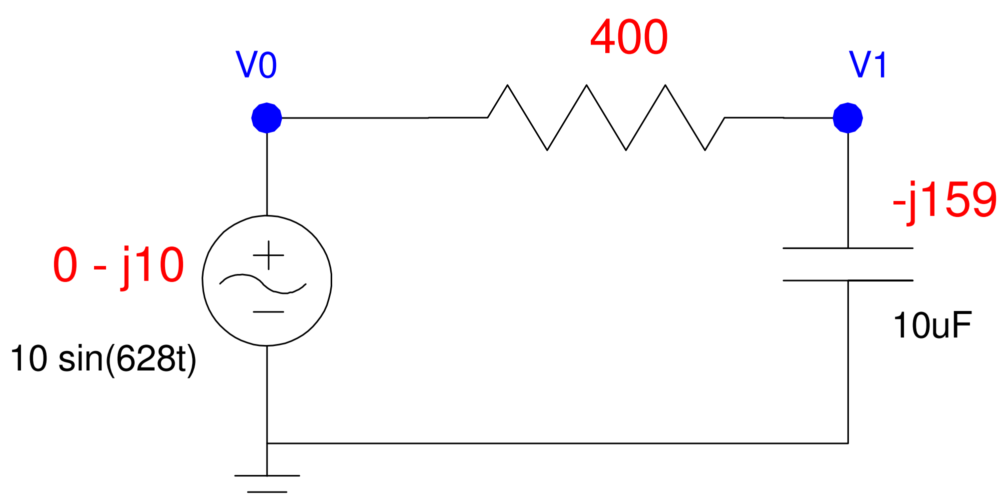
$$V_1 = -3.436 - j1.368$$

$$v_1(t) = -3.436\cos(628t) + 1.368\sin(628t)$$

In polar form

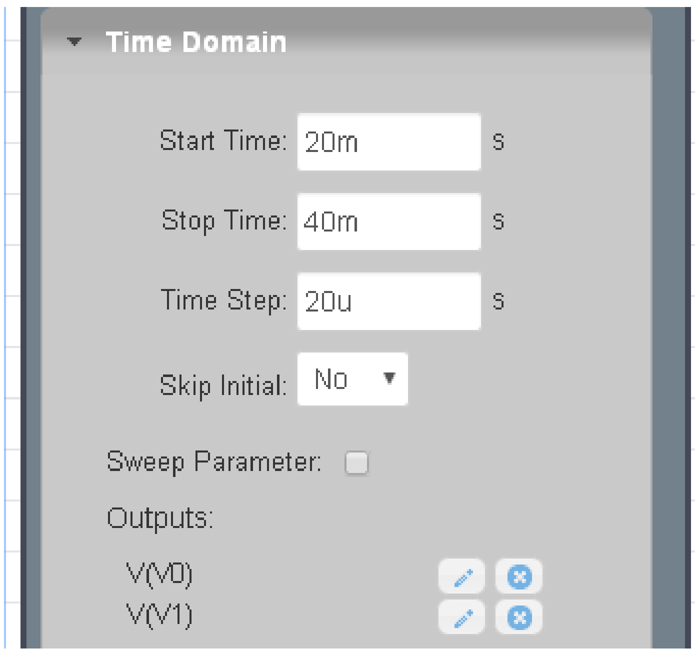
$$V_1 = 3.694\angle -158.3^0$$

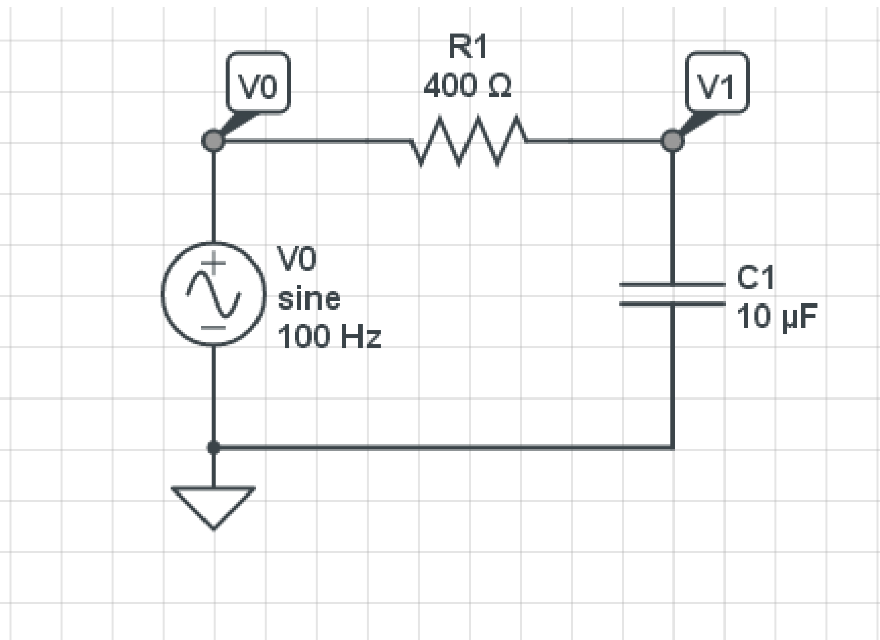
$$v_1(t) = 3.694\cos(628t - 158.3^0)$$



Check in CircuitLab

- Time-domain response
- 20ms (2 cycles)

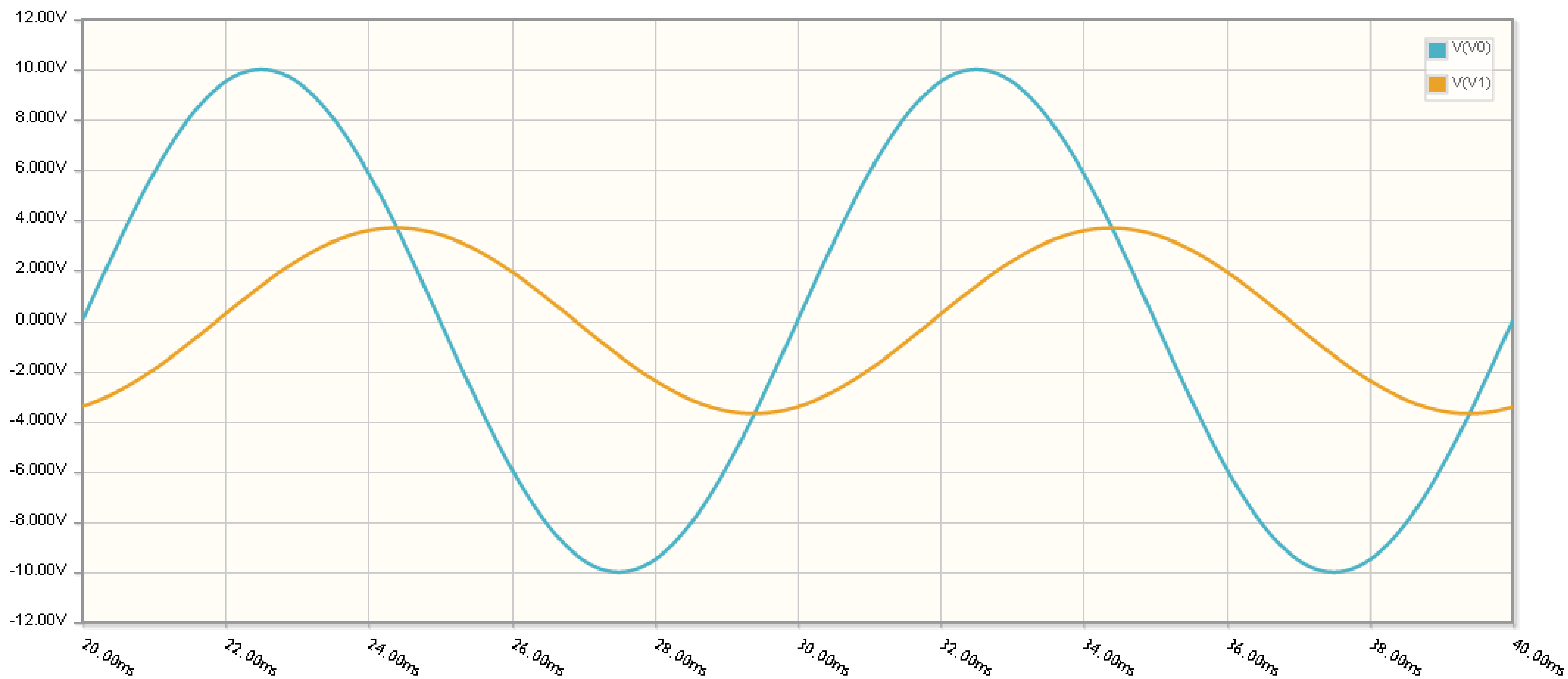




Note from the CircuitLab plot, matches our calculations:

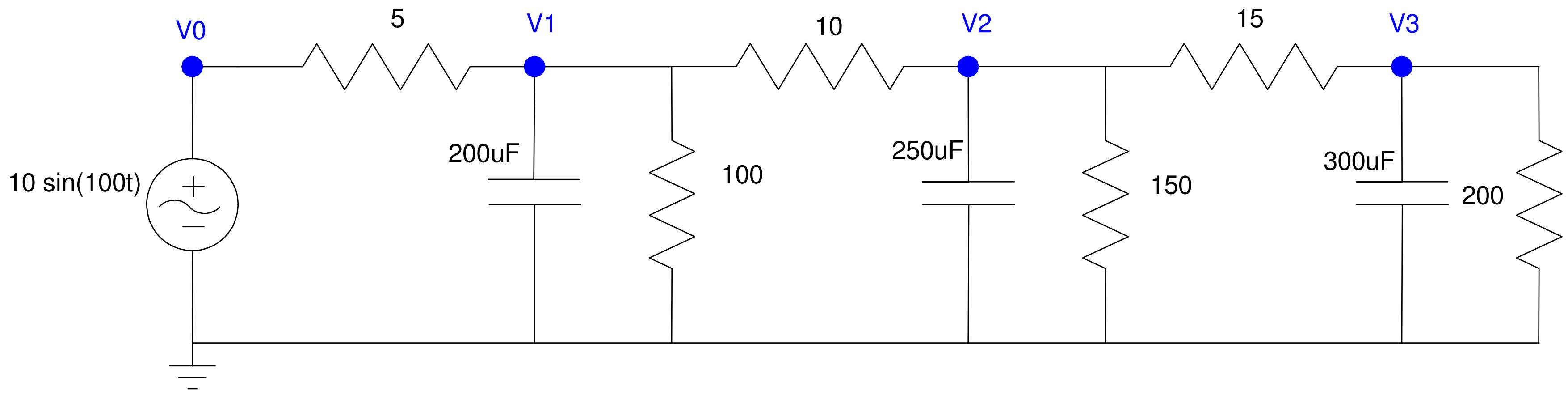
- The peak for V1 (orange) is 3.694V
- V1 is delayed from V0 by 4.5ms

$$\theta = -\left(\frac{\text{delay (ms)}}{\text{period (ms)}}\right)360^0 = -\left(\frac{4.5ms}{10ms}\right)360^0 = -162^0$$



Example 2: 3-Stage RC Circuit

Find the voltages for the following circuit



Step 1: Convert to phasors

$V_0 = 10\sin(100t)$

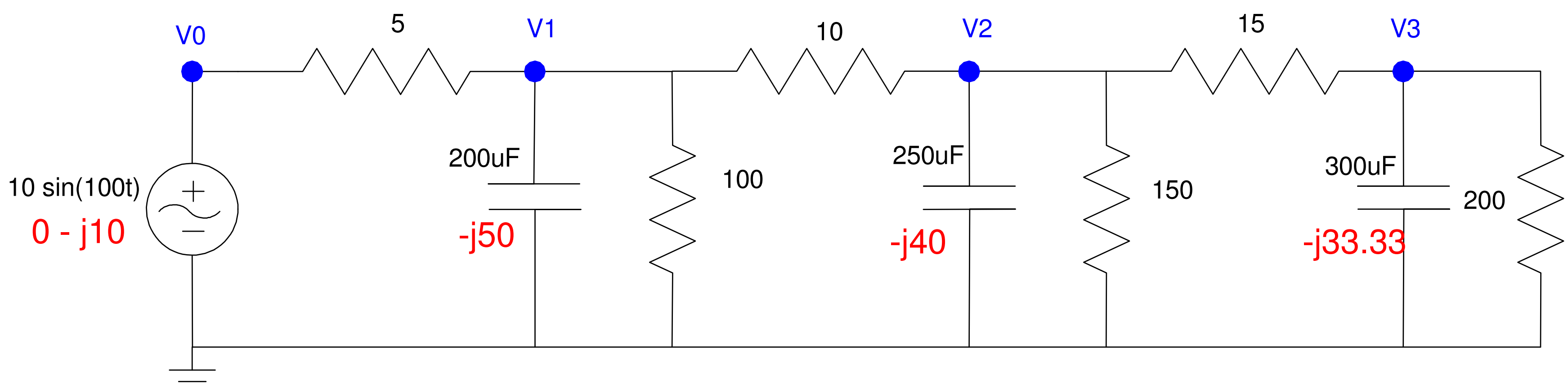
$$V_0 = 0 - j10$$

$$\omega = 100$$

0.01F: $Z_c = \frac{1}{j\omega C} = -j50\Omega$

0.02F: $Z_c = \frac{1}{j\omega C} = -j40\Omega$

0.03F: $Z_c = \frac{1}{j\omega C} = -j33.33\Omega$



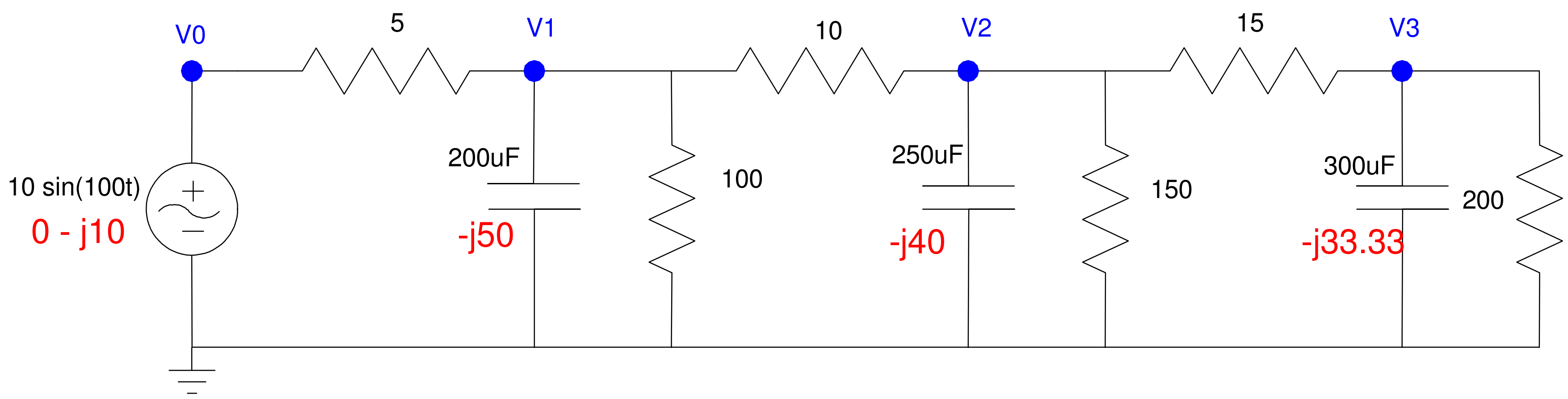
Step 2: Write N equations for N unknowns

V0: $V_0 = 0 - j10$

V1: $\left(\frac{V_1 - V_0}{5}\right) + \left(\frac{V_1}{100}\right) + \left(\frac{V_1}{-j50}\right) + \left(\frac{V_1 - V_2}{10}\right) = 0$

V2: $\left(\frac{V_2 - V_1}{10}\right) + \left(\frac{V_2}{150}\right) + \left(\frac{V_2}{-j40}\right) + \left(\frac{V_2 - V_3}{15}\right) = 0$

V3: $\left(\frac{V_3 - V_2}{15}\right) + \left(\frac{V_3}{200}\right) + \left(\frac{V_3}{-j33.33}\right) = 0$



Step 3: Solve. First, group terms

$$V_0 = -j10$$

$$-\left(\frac{1}{5}\right)V_0 + \left(\frac{1}{5} + \frac{1}{100} + \frac{1}{-j50} + \frac{1}{10}\right)V_1 + \left(\frac{-1}{10}\right)V_2 = 0$$

$$\left(\frac{-1}{10}\right)V_1 + \left(\frac{1}{10} + \frac{1}{150} + \frac{1}{-j40} + \frac{1}{15}\right)V_2 + \left(\frac{-1}{15}\right)V_3 = 0$$

$$\left(\frac{-1}{15}\right)V_2 + \left(\frac{1}{15} + \frac{1}{200} + \frac{1}{-j33.33}\right)V_3 = 0$$

Place in matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \left(\frac{-1}{5}\right) & \left(\frac{1}{5} + \frac{1}{100} + \frac{1}{-j50} + \frac{1}{10}\right) & \left(\frac{-1}{10}\right) & 0 \\ 0 & \left(\frac{-1}{10}\right) & \left(\frac{1}{10} + \frac{1}{150} + \frac{1}{-j40} + \frac{1}{15}\right) & \left(\frac{-1}{15}\right) \\ 0 & 0 & \left(\frac{-1}{15}\right) & \left(\frac{1}{15} + \frac{1}{200} + \frac{1}{-j33.33}\right) \end{bmatrix} \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} -j10 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Put into MATLAB and solve

```
a1 = [1,0,0,0];
a2 = [-1/5,1/5+1/100+1/(-j*50)+1/10,-1/10,0];
a3 = [0,-1/10,1/10+1/150+1/(-j*40)+1/15,-1/15];
a4 = [0,0,-1/15,1/15+1/200+1/(-j*33.33)];
A = [a1;a2;a3;a4]

   1.0000                   0                   0                   0
  -0.2000         0.3100 + 0.0200i  -0.1000                   0
        0        -0.1000             0.1733 + 0.0250i  -0.0667
        0                   0       -0.0667             0.0717 + 0.0300i

B = [-j*10;0;0;0];
V = inv(A)*B
```

V0   0 -10.0000i
V1 -1.6314 - 8.0724i
V2 -3.4430 - 5.3506i
V3 -4.4982 - 3.0942i

meaning

$$V_0 = 10\sin(100t)$$

$$V_1 = -1.6314\cos(100t) + 8.0742\sin(100t)$$

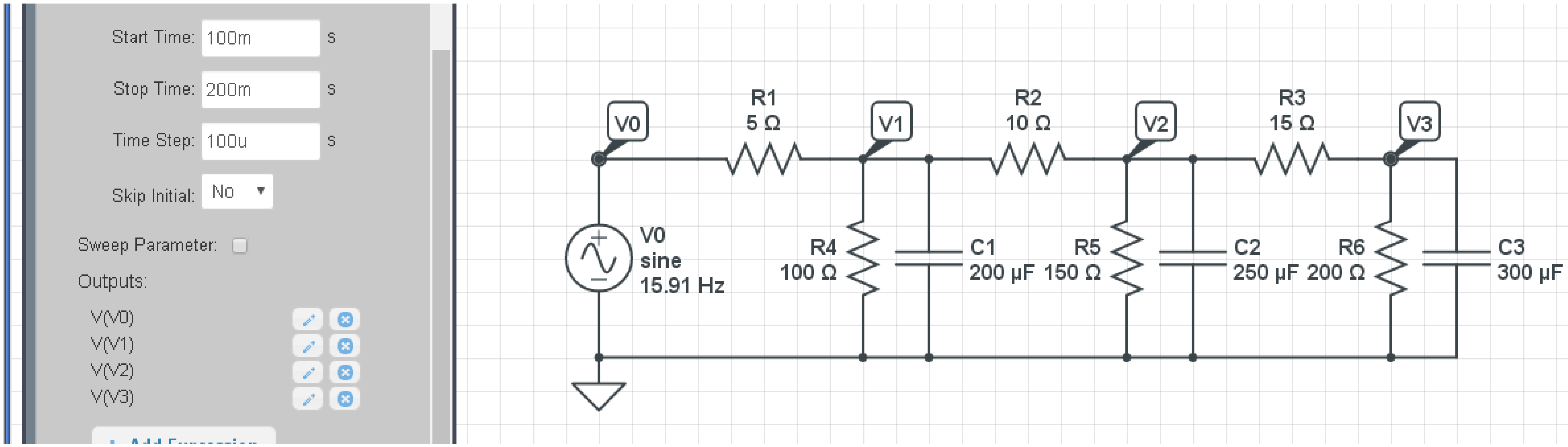
$$V_2 = -3.4430\cos(100t) + 5.5306\sin(100t)$$

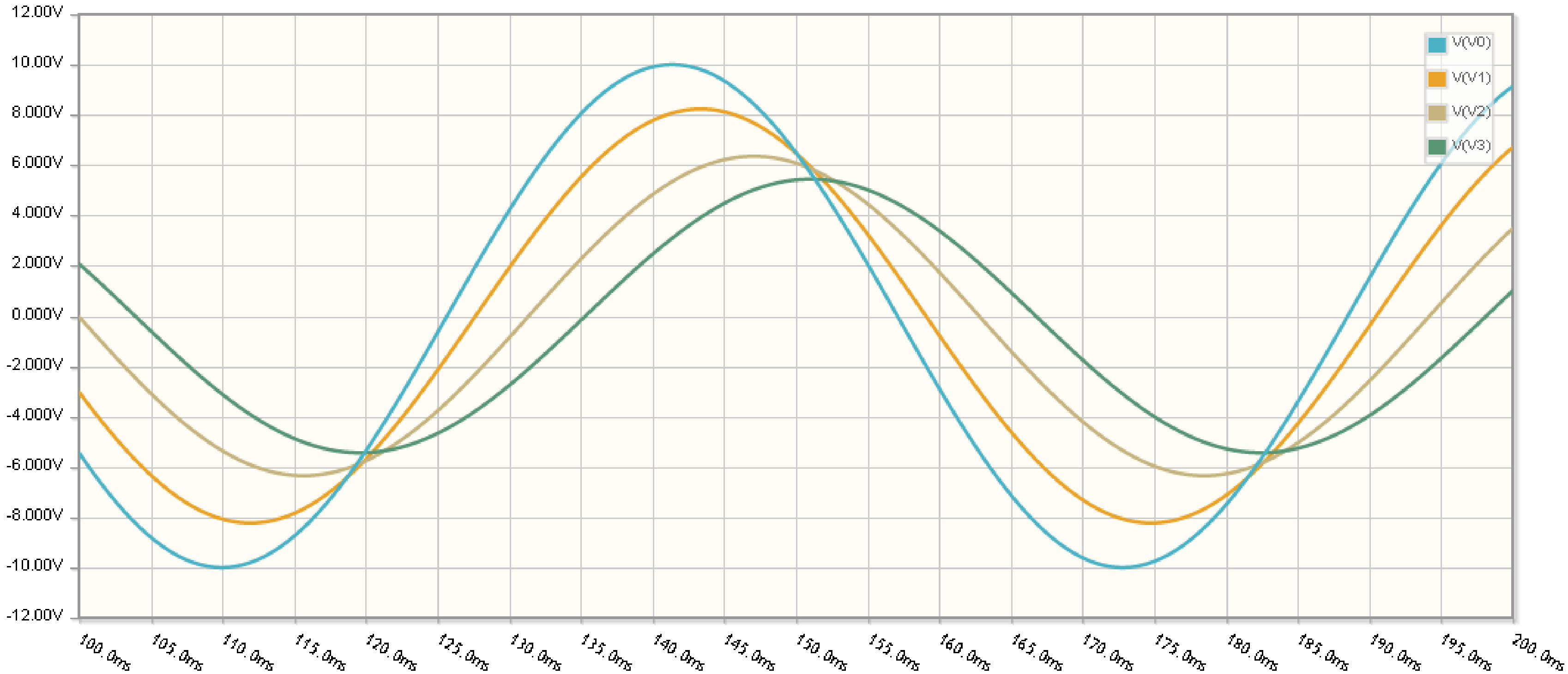
$$V_3 = -4.4982\cos(100t) - 3.0942\sin(100t)$$

The magnitude of each voltage is:

```
abs(V)

   10.0000
    8.2356
    6.3627
    5.4596
```

CircuitLab Simulation





	|Vin|	|V1|	|V2|	|V3|
Calculated	10.0000V	8.2356V	6.3627V	5.4596V
CircuitLab	10.00V	8.231V	6.348V	5.435V

Summary

Real numbers work well when analyzing DC circuits

Complex numbers work well when analyzing AC circuits

Everything that we did with DC circuits with AC circuits - only you wind up with complex numbers

For voltages

- The real part represents the cosine term
- The complex part is represents the minus-sine term

For impedances

- Resistors are real (R)
- Inductors are +jX (Z = jwL)
- Capacitors are -jX (Z = 1/(jwC))