Complex Numbers

ECE 111 Introduction to ECE

Jake Glower - Week #12

Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

Topics

- Complex Numbers
- Complex Numbers on an HP42 Calculator

Objectives

- Be able to add & subtract complex numbers
- Be able to multiply and divide complex numbers
- Be able to switch between rectangular and polar representation for complex numbers
- Be able to do partial fraction expansion with complex numbers

Introduction

Numbers matter.

- Numbers make a difference.
- The existence of mathematical oddities can determine the fate of empires.

The Number Zero:

Why have the number zero?

- Not necessary
- Rome ran a vast empire without the number zero

Roman Numbers:

Romans represented numbers with letters:

- I = 1
- V = 5
- X = 10
- L = 50
- C = 100
- IX = 9 (10 - 1)
- XC = 90 (100 - 10)

Example:

- 27 = XXVII
- 109 = CIX

Addition, Subtration, Multiplication, & Division

Totally possible with Roman numbers

- Rome ran a vast empire
- You know that addition, subtration, multiplication, and division happened

It's not easy with Roman numbers

Addition	Multiplication
XXVII	XXVII
+ CIX	* CIX
= ?	= ?

Arabic Numbers

In contrast, the Arabic number system uses place holders

- Each digit represents an increasing power of ten
- With this number system, you *need* the number zero.

Example:

$$106 = 1 \times 100 + 0 \times 10 + 6 \times 1$$

106 is very different than 16.

With Arabic numbers,

- Addition and subtraction become much easier
- Multiplication and division remains a challenge
- But it is far easier than with Roman numbers.

Negative Numbers:

Why have negative numbers?

- What does negative one apple mean?
- Why allow this to be represented?

Negative numbers are not necessary

- They do make some calculations easier, however.

Example: Voltage Nodes

- The sum of the current from a node must be zero
- If you don't mind negative numbers, you don't need to know which direction the current is flowing before you write the equations

Accounting & Double-Entry Bookkeeping System

Florence was a world power in the 13th century.

- A remarkable feat considering that Florence is a small city in Italy
- Competing against much larger countries such as Spain, France, and England.

Republic of Florence: a small city state competing for European dominance (Wikipedia)

One reason for this was the invention of the double-entry bookkeeping system.

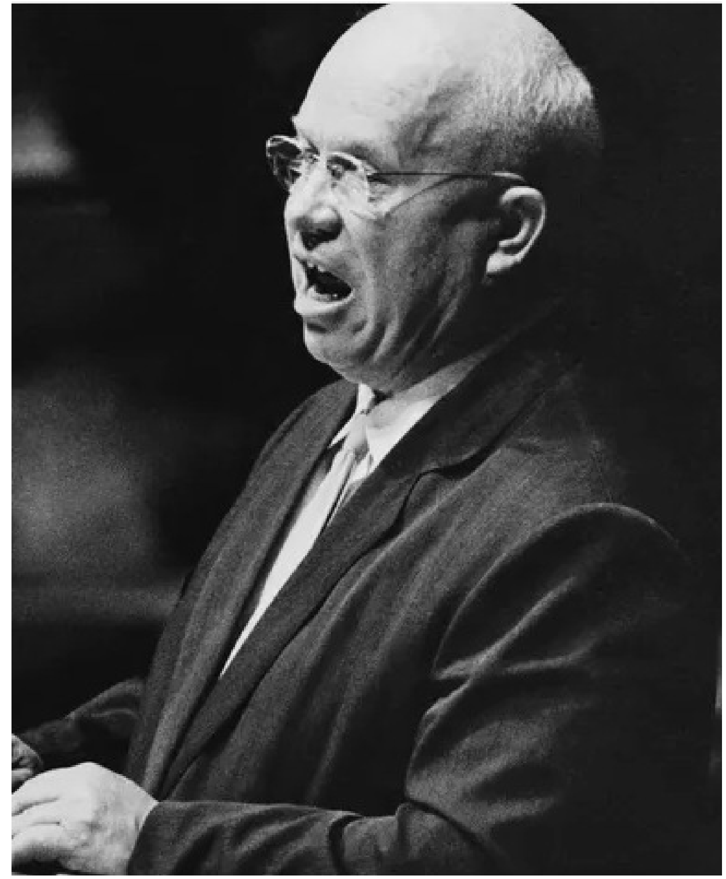
- Keep track of debits (negative profit)
- Keep track of credits (positive profit)
- Credits minus debits = profit

With this, Florentine merchants could keep track of which ventures were profitable and which were not.

- Everyone else just looked at the return at the end of a venture
- Ignoring how much money you sunk into it over the course of time

"We will bury you"

- Nikita Khrushchev



Accounting can decide the fate of empires

In the 1960's, the Soviet Union *thought* their economy was growing 20% per year

- vs. 4% for the United States

At that rate, by 2000 the Soviet Union would crush the U.S. ecomomically

- Exponential growth is a powerful thing...

Actually, the Soviet economy was shrinking 1-2% per year

Eventually, the size of the goverment (based upon 20% growth) could not be supportedAccounting & Empires

Accounting can determine the fate of empires

Complex Numbers:

Real numbers work well for DC circuits

- 1st half of EE 206 Circuits I
- Voltages can be expressed by a real number
- Currents can be expressed with real numbers, and
- Resistance's can be expressed with real numbers.

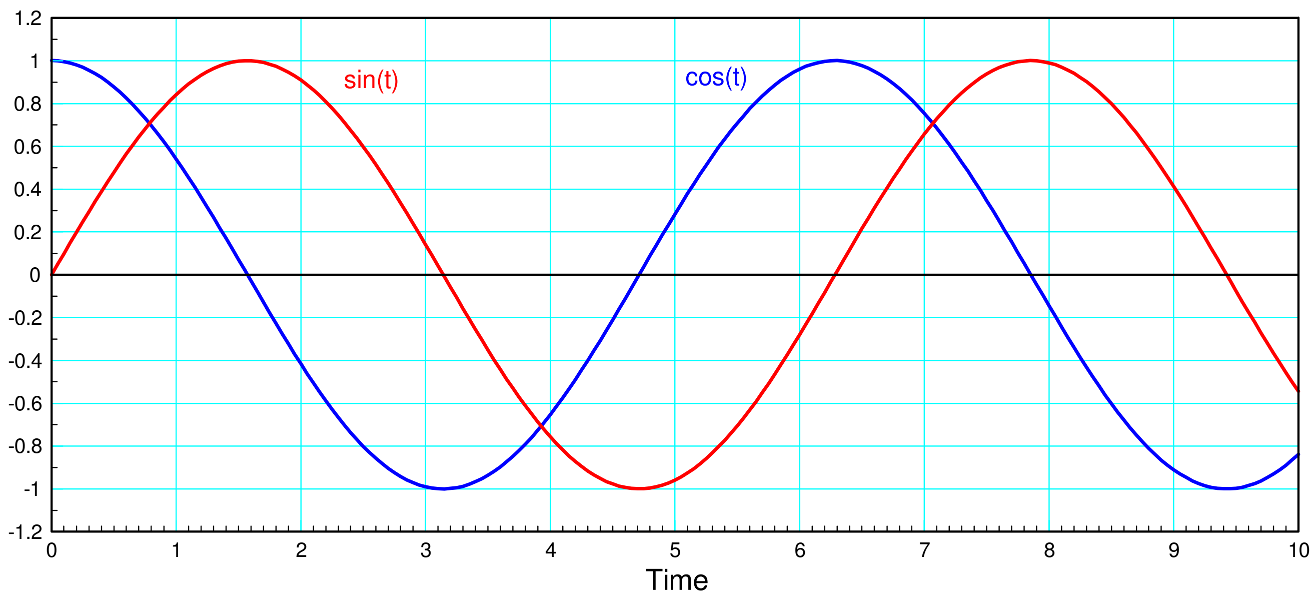
Real numbers have problems when working with AC circuits

- 2nd half of EE 206 Circuits I
- Rest of ECE curriculum

With AC signals, you have two terms:

- sin(wt)
- cos(wt)

You likewise need two numbers to represent any given voltage or current when dealing with AC signals



sin(t) and cos(t). Note that the period is 2 pi

A generalized sine wave can be represented as

$$x(t) = a\cos(\omega t) + b\sin(\omega t)$$

or

$$x(t) = r\cos(\omega t + \theta)$$

(more on this later).

What this means is, unlike DC, you need three parameters to define a sinusoid:

- The frequency (w), and
- The cosine and sine terms (a, and b), or
- The amplitude and phase shift (r and θ)

Frequency isn't a problem:

- If you have a linear circuit,
- The frequency doesn't change
- All signals will be the same frequency as the input.

Voltage and current are a problem

- You need a way to represent two parameters:
  - The magnitude of the sine() and cosine() terms in rectangular form, or
  - The amplitude and angle in polar form

Complex numbers provide you with those two degrees of freedom

- Any time you're working with DC signals, real numbers suffice
- Any time you're working with AC signals, complex numbers are used

In short, ECE majors use complex numbers - probably more than any other major.

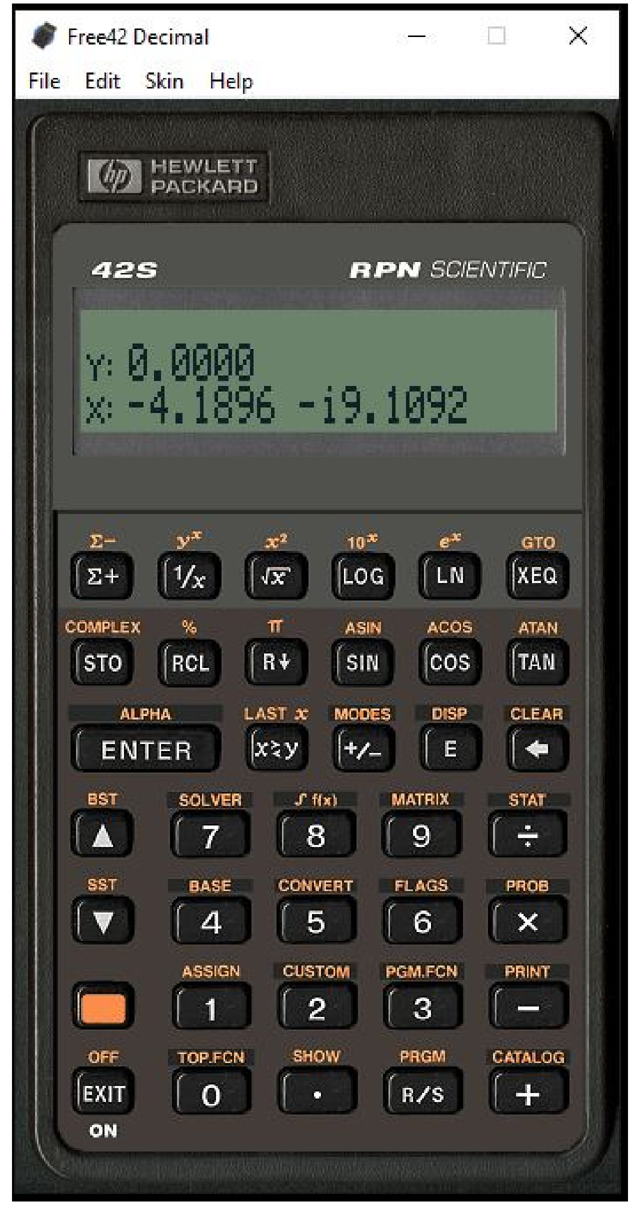
Fortunately, Matlab (and HP calculators) have no problem working with complex numbers.

Free42 is a free app for your cell phone

- It emulates the HP42 calculator
- The *best* calculator ever made for ECE majors
- Deals with complex numbers with ease
- Worth about 10 points on midterms in ECE

It has a learning curve though

- Don't try to learn how to use an HP during a midterm



Complex Numbers

Unlike real numbers, complex numbers have two terms. This allows us to represent the cosine and sine terms for a sinusoid with a single (albeit complex) number.

The basic idea behind complex numbers is to define a term, j, as[1]

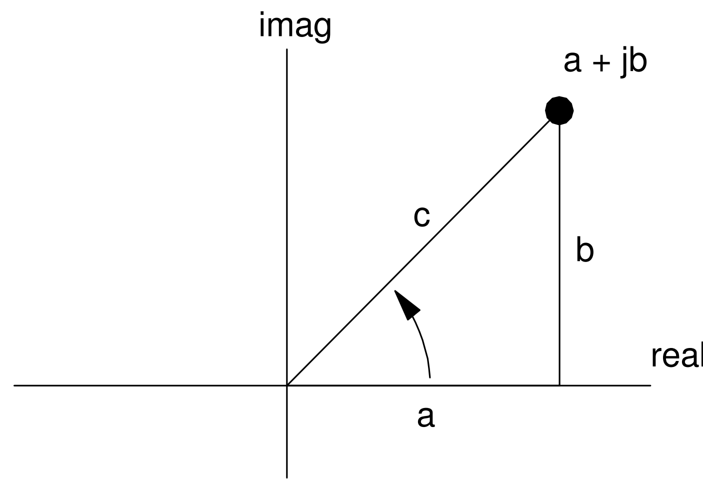
$$j \equiv \sqrt{-1}$$

Any given number can then have a real and a complex part

$$x = a + jb$$

You can express this number in rectangular form (a + jb) or polar form

$$x = c\angle\theta$$



[1] Math majors call this term i for imaginary. In ECE, i means current, so we use the letter j to represent the complex part of a number.

Note: The polar form is shorthand notation and actually means

$$c\angle\theta \equiv c \cdot e^{j\theta}$$

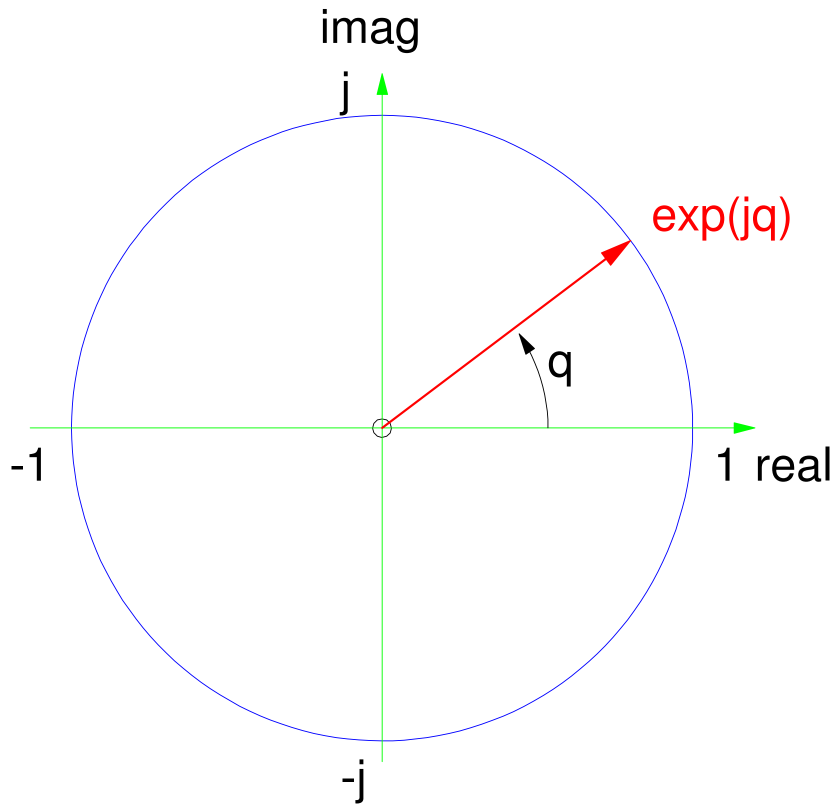
The complex exponential has two terms

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

This leads to Euler's identity

$$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$$



Proof: Substitute for the complex exponential

$$\left(\frac{e^{j\theta}+e^{-j\theta}}{2}\right) = \frac{1}{2}((\cos\theta + j\sin\theta) + (\cos(-\theta) + j\sin(-\theta)))$$

Using some trig identities

$$\cos(-\theta) = \cos\theta$$

$$\sin(-\theta) = -\sin\theta$$

gives

$$= \left(\frac{1}{2}\right)(\cos\theta + j\sin\theta + \cos\theta - j\sin\theta)$$

$$= \frac{1}{2}(2\cos\theta)$$

$$= \cos\theta$$

The proof for sin(x) is similar

Inputting Complex Numbers into Matlab

With Matlab, the default value of j is $\sqrt{-1}$.

If you redefine j, this no longer holds, but you can restore this as

```
>> j = sqrt(-1)
```

To input a number into Matlab in rectangular form, simply use the j variable

```
>> A = 2 + j*3

A =   2.0000 + 3.0000i
```

You can also input a variable in polar form. $B = 3\angle 1.5$ is input as

```
>> B = 3 * exp(j*1.5)

B =   0.2122 + 2.9925i
```

(note: Matlab uses radians for its angle units).

The default display in Matlab is rectangular units. To convert to polar, use the abs and angle

```
>> abs(B)

ans =     3

>> angle(B)

ans =    1.5000
```

Inputting Complex Numbers into an HP42

This uses the complex key. In rectangular mode (yellow - MODES - RECT)

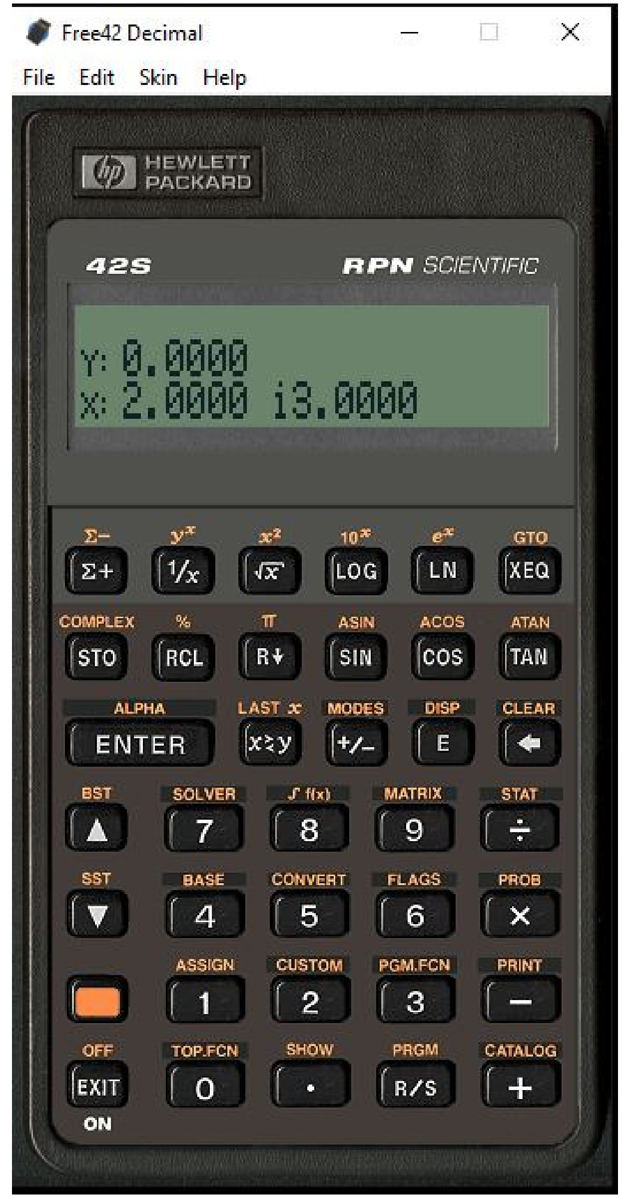
- the y register becomes the real part
- the x register becomes the complex part.

To input the number 2 + j3, press

```
MODES
RECT
2
enter
3
complex
```

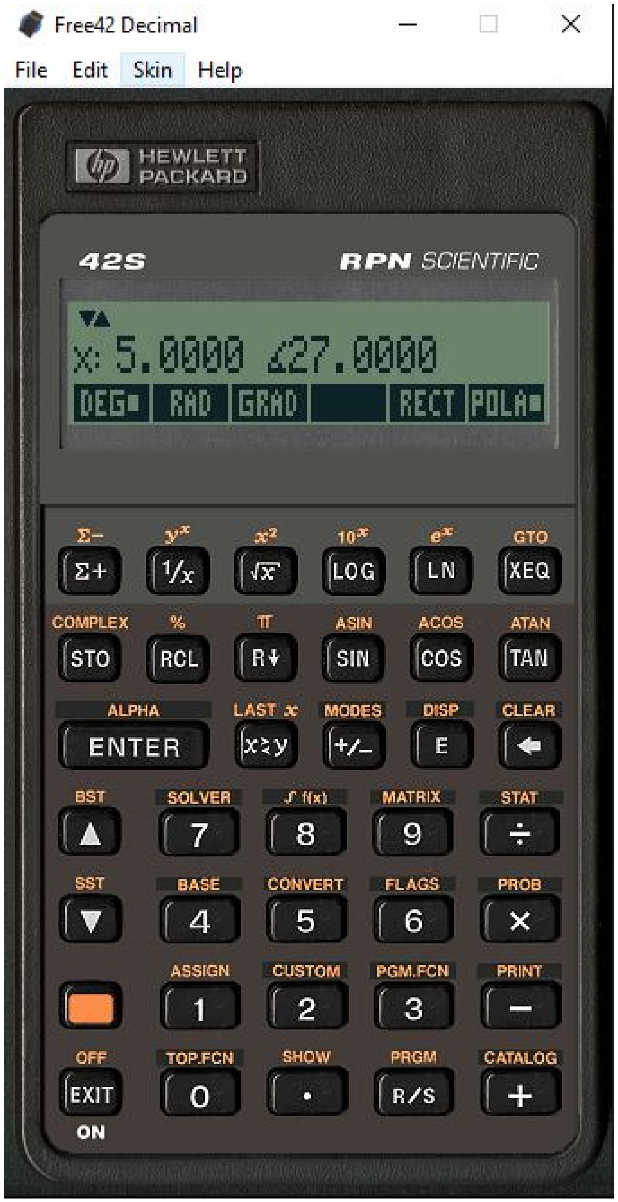
In polar mode (yellow - MODES - POLAR)

- the y register becomes the magnitude
- the x register becomes the angle (in the current units degrees / rad / grad)



To input the number $5\angle 27^0$

```
MODES
POLAR
5
enter
27
complex
```



Addition, Subtraction, Multiplication, and Division

With complex numbers, you can add, subtract, multiply, and divide just like real numbers:

Addition: For addition

- The real parts add, and
- The complex parts add.

Example:

$$(a_1 + jb_1) + (a_2 + jb_2) = (a_1 + a_2) + j(b_1 + b_2)$$

Subtraction: Again, the real parts subtract and the complex parts subtract

$$(a_1 + jb_1) - (a_2 + jb_2) = (a_1 - a_2) + j(b_1 - b_2)$$

Note: Addition and subtraction also work in polar form.

- This requires a polar to rectangular conversion:

$$r_1\angle\theta_1 + r_2\angle\theta_2 = (r_1\cos\theta_1) + j(r_1\sin\theta_1) + (r_2\cos\theta_2) + j(r_2\sin\theta_2)$$
$$= (r_1\cos\theta_1 + r_2\cos\theta_2) + j(r_1\sin\theta_1 + r_2\sin\theta_2)$$

Moral: Addition and subtraction is easier in rectangular form. Or use a calculator that can add and subtract complex numbers.

Multiplication: Multiplication is a little trickier, but the result is a complex number.

$$(a_1 + jb_1)(a_2 + jb_2) = a_1 a_2 + ja_1 b_2 + ja_2 b_1 + j^2 b_1 b_2$$

Note that $j^2 = -1$

$$(a_1 + jb_1)(a_2 + jb_2) = (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + a_2 b_1)$$

Polar form actually works better for multiplication

$$r_1 \angle \theta_1 \cdot r_2 \angle \theta_2 = r_1 e^{j\theta_1} \cdot r_2 e^{j\theta_2}$$

$$= r_1 r_2 e^{j\theta_1} e^{j\theta_2}$$

Using the property

$$e^a e^b = e^{a+b}$$

gives

$$r_1 \angle \theta_1 \cdot r_2 \angle \theta_2 = r_1 r_2 e^{j\theta_1 + j\theta_2}$$

$$r_1 \angle \theta_1 \cdot r_2 \angle \theta_2 = r_1 r_2 \angle (\theta_1 + \theta_2)$$

When you multiply complex numbers

- The magnitude multiplies and
- The angles add

Complex Conjugate: The complex conjugate of a complex number is

$$(a+jb)^* \equiv a-jb$$

The complex conjugate has the property that the product of a complex number with its complex conjugate is a real number, equal to the magnitude squared:

$$(a+jb)\cdot(a-jb) = a^2+b^2$$

Division: Division also results in a complex number but takes even more computations. It uses the complex conjugate of the denominator:

$$\begin{aligned}
\left(\frac{a_1+jb_1}{a_2+jb_2}\right) &= \left(\frac{a_1+jb_1}{a_2+jb_2}\right)\left(\frac{a_2-jb_2}{a_2-jb_2}\right) \\
&= \left(\frac{(a_1a_2+b_1b_2)+j(-a_1b_2+a_2b_1)}{a_2^2+b_2^2}\right) \\
&= \left(\frac{a_1a_2+b_1b_2}{a_2^2+b_2^2}\right) + j\left(\frac{-a_1b_2+a_2b_1}{a_2^2+b_2^2}\right)
\end{aligned}$$

Polar form is again simpler for division

$$\left(\frac{r_1\angle\theta_1}{r_2\angle\theta_2}\right) = \left(\frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}}\right)$$

$$= \left(\frac{r_1}{r_2}\right) e^{j\theta_1} e^{-j\theta_2}$$

$$= \left(\frac{r_1}{r_2}\right) e^{j\theta_1 - k\theta_2}$$

$$= \left(\frac{r_1}{r_2}\right) \angle(\theta_1 - \theta_2)$$

The division of complex numbers is

- The ratio of the magnitudes and
- The difference in the angles.

Handout: Solve the following problems

Determine the result of the following operations with complex numbers

```
   2 + j3       2 + j3          2 + j3          2 + j3
+  9 + j4   -   9 + j4      *   9 + j4       ------------
----------- ----------     -----------          9 + j4
```

Sample Problems

Problem 1: Find y

$$y = \left(\frac{(2+j3)(4+j5)+(6+j7)}{8+j9} \right)$$

By Hand: (pretty painful)

$$y = \left(\frac{(2+j3)(4+j5)+(6+j7)}{8+j9}\right)$$

$$= \left(\frac{((8-15)+j(12+10))+(6+j7)}{(8+j9)}\right)$$

$$= \left(\frac{(-7+j22)+(6+j7)}{(8+j9)}\right)$$

$$= \left(\frac{-1+j29}{8+j9}\right)\left(\frac{8-j9}{8-j9}\right)$$

$$= \left(\frac{(-8+261)+j(232+9)}{64+81}\right)$$

$$= \left(\frac{253+j241}{145}\right)$$

$$= \left(\frac{253}{145}\right)+j\left(\frac{241}{145}\right)$$

$$= 1.7448 + j1.6621$$

Using Matlab

$$y = \left(\frac{(2+j3)(4+j5)+(6+j7)}{8+j9} \right)$$

```
>> y = ( (2+j*3)*(4+j*5) + (6+j*7) ) / (8 + j*9)
```

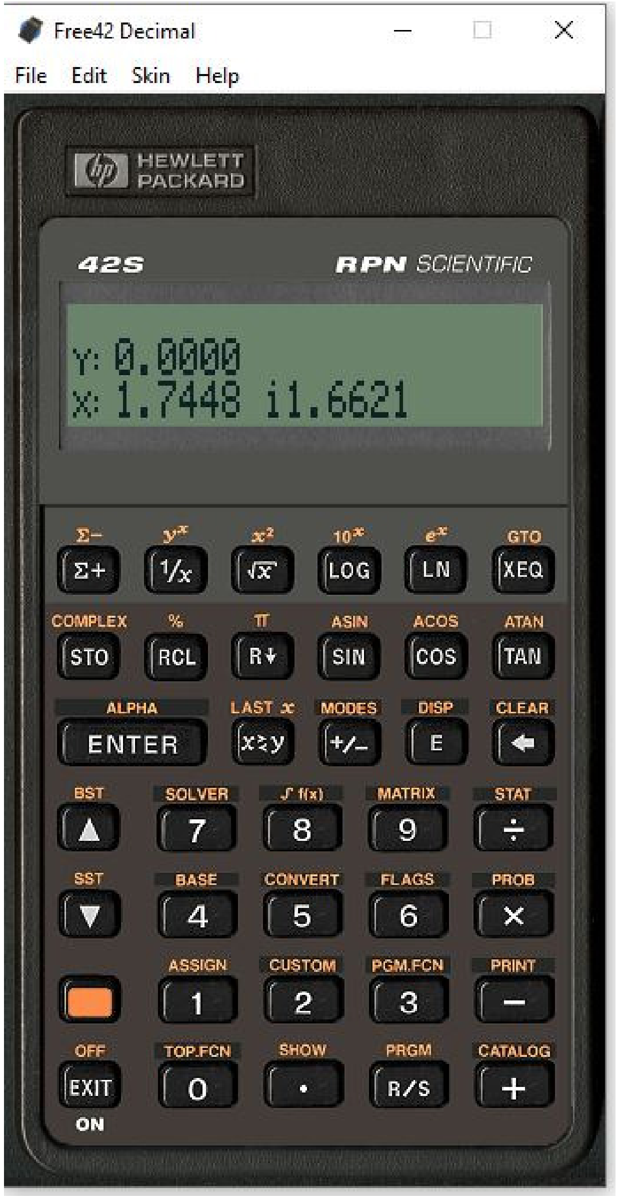
`y =   1.7448 + 1.6621i`

Using an HP42

$$y = \left(\frac{(2+j3)(4+j5)+(6+j7)}{8+j9}\right)$$

```
2
enter
3
complex
4
enter
6
complex
*
6
enter
7
complex
+
8
enter
9
complex
/
```

ans = 1.7448 + j1.6621



Partial Fraction Expansion with Real Poles

A common problem in ECE is to expand a function by its roots. For example, find

$$\left(\frac{2x+3}{(x+1)(x+2)(x+3)}\right) = \left(\frac{a}{x+1}\right) + \left(\frac{b}{x+2}\right) + \left(\frac{c}{x+3}\right)$$

Solution #1: (the hard way) Place the right side over a common denominator and match coefficients.

$$= \left(\frac{a}{x+1}\right)\left(\frac{(x+2)(x+3)}{(x+2)(x+3)}\right) + \left(\frac{b}{x+2}\right)\left(\frac{(x+1)(x+3)}{(x+1)(x+3)}\right) + \left(\frac{c}{x+3}\right)\left(\frac{(x+1)(x+2)}{(x+1)(x+2)}\right)$$

This places all terms over a common denominator. The numerator is then

$$2x+3 = a(x+2)(x+3) + b(x+1)(x+3) + c(x+1)(x+2)$$

$$2x+3 = a(x^2+5x+6) + b(x^2+4x+3) + c(x^2+3x+2)$$

This gives three equations for three unknowns.

Matching the x^2 terms:

$$0 = a+b+c$$

x^1 terms:

$$2 = 5a+4b+3c$$

x^0 terms:

$$3 = 6a+3b+2c$$

Place in matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 5 & 4 & 3 \\ 6 & 3 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$$

Solve in Matlab

```
>> A = [1,1,1;5,4,3;6,3,2]
     1      1      1
     5      4      3
     6      3      2

>> B = [0;2;3]
     0
     2
     3
```

```
>> inv(A)*B

a    0.5000
b    1.0000
c   -1.5000
```

so

$$\left(\frac{2x+3}{(x+1)(x+2)(x+3)}\right) = \left(\frac{0.5}{x+1}\right) + \left(\frac{1}{x+2}\right) + \left(\frac{-1.5}{x+3}\right)$$

Solution #2: (cover-up method).

Equals is a powerful symbol: it means both sides are identical everywhere.

- The right side blows up (goes to infinity) near x = {-1, -2, -3}. The left side has to match.

Near x = -1, only the first term matters since it's going to infinity while the other terms are finite. So

$$\lim_{x\to -1}\left(\frac{2x+3}{(x+1)(x+2)(x+3)}\right) = \lim_{x\to -1}\left(\frac{a}{x+1}\right)$$

Cancel (cover up) the (x+1) term and evaluate

$$a = \left(\frac{2x+3}{(x+2)(x+3)}\right)_{x=-1} = 0.5$$

Near x = -2, only the second term (b) matters on the right.

$$\lim_{x\to -2}\left(\frac{2x+3}{(x+1)(x+2)(x+3)}\right)=\lim_{x\to -2}\left(\frac{b}{x+2}\right)$$

Cancel (cover up) the (x+1) term and evaluate

$$b=\left(\frac{2x+3}{(x+1)(x+3)}\right)_{x=-2}=1$$

Near x = -3, only the third term (c) matters:

$$\lim_{x\to -3}\left(\frac{2x+3}{(x+1)(x+2)(x+3)}\right)=\lim_{x\to -3}\left(\frac{c}{x+3}\right)$$

$$c=\left(\frac{2x+3}{(x+1)(x+2)}\right)_{x=-3}=-1.5$$

Either method works - the cover-up method is a lot easier.

The cover-up method also works in Matlab.

- Take a number close to the point you're evaluating (perturb by 1e-9)
- Solve for {a, b, c}

```
>> x = -1 + 1e-9;
>> a = (2*x + 3)/( (x+1)*(x+2)*(x+3) ) * (x+1)

a =    0.5000

>> x = -2 + 1e-9;
>> b = (2*x + 3)/( (x+1)*(x+2)*(x+3) ) * (x+2)

b =    1.0000

>> x = -3 + 1e-9;
>> c = (2*x + 3)/( (x+1)*(x+2)*(x+3) ) * (x+3)

c =   -1.5000
```

Handout: Find the partial fraction expansion

$$\left(\frac{10(x+2)}{(x+5)x(x+10)}\right) = \left(\frac{a}{x+5}\right) + \left(\frac{b}{x}\right) + \left(\frac{c}{x+10}\right)$$

Partial Fraction with Complex Numbers

Placing all terms over a common denominator works, but is that much harder with complex numbers. The cover-up method is the same either way.

Example: Determine {a, b, c}

$$\left(\frac{5x+7}{(x+1+j3)(x+1-j3)(x+5)}\right) = \left(\frac{a}{x+1+j3}\right) + \left(\frac{b}{x+1-j3}\right) + \left(\frac{c}{x+5}\right)$$

Solving

$$a = \left(\frac{5x+7}{(x+1-j3)(x+5)}\right)_{x=-1-j3} = 0.3600 + j0.3533$$

$$b = \left(\frac{5x+7}{(x+1+j3)(x+5)}\right)_{x=-1+j3} = 0.3600 - j0.3533$$

$$c = \left(\frac{5x+7}{(x+1+j3)(x+1-j3)}\right)_{x=-5} = -0.7200$$

Solving using Matlab

```
>> x = -1 - j*3 + 1e-9;
>> a = (5*x+7) / ( (x+1+j*3)*(x+1-j*3)*(x+5) ) * (x+1+j*3)

a =   0.3600 + 0.3533i

>> x = -1 + j*3 + 1e-9;
>> a = (5*x+7) / ( (x+1+j*3)*(x+1-j*3)*(x+5) ) * (x+1-j*3)

a =   0.3600 - 0.3533i

>> x = -5 + 1e-9;
>> a = (5*x+7) / ( (x+1+j*3)*(x+1-j*3)*(x+5) ) * (x+5)

a =   -0.7200
```

More Fun with Complex Numbers

Note: These use the properties

$$\ln(e^x) = x$$

$$e^{\ln(x)} = x$$

$$(e^a)^b = e^{ab}$$

Example 1: Find y:

$$y = (2 + j3)^{(4+j5)}$$

Solution: Convert to polar form (using radians)

$$2 + j3 = 3.6056\angle 0.9828$$

$$= e^{\ln(3.6056)} \cdot e^{j0.9828}$$

$$= e^{1.2825 + j0.9828}$$

Raise to the power

$$= (e^{1.2825 + j0.9828})^{(4+j5)}$$

$$= e^{(0.2159 + j10.3453)}$$

Separate

$$= e^{0.2159} e^{j10.3435}$$

$$= 1.2410 \cdot (\cos(10.3425) + j\sin(10.3435))$$

$$= 1.2410 \cdot (-0.6068 - j0.7942)$$

$$= -0.7530 - j0.9864$$

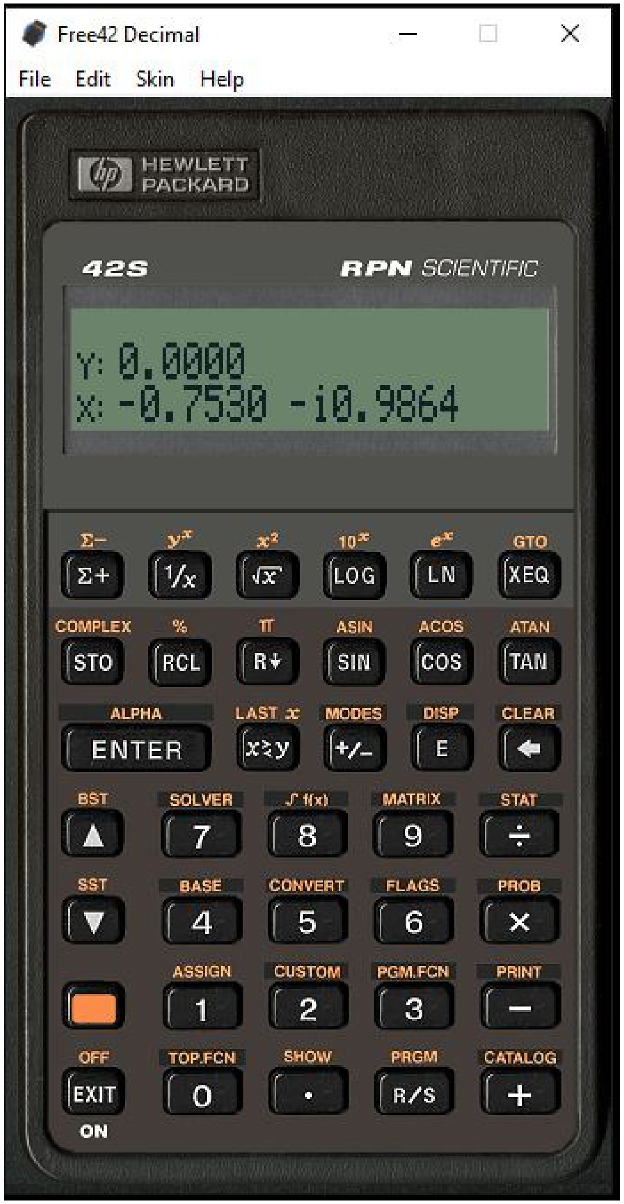
Check in Matlab

```
>> y = (2 + j*3) ^ (4 + j*5)

y =  -0.7530 - 0.9864i
```

Note: Matlab was *way* easier.

Check on an HP42:

```
2
enter
3
complex
4
enter
5
complex
y^x

ans = -0.7530 - j0.9864
```



Example 2: Find y:

$$y = \cos(2 + j3)$$

Use Euler's identity

$$\cos(2 + j3) = \left(\frac{1}{2}\right)\left(e^{(2+j3)} + e^{j(2+j3)}\right)$$

$$= \left(\frac{1}{2}\right)((e^{2}e^{j3}) + (e^{j2}e^{-3}))$$

$$= \left(\frac{1}{2}\right)(e^{2} \cdot (\cos(3) + j\sin(3)) + e^{-3} \cdot (\cos(2) + j\sin(2))$$

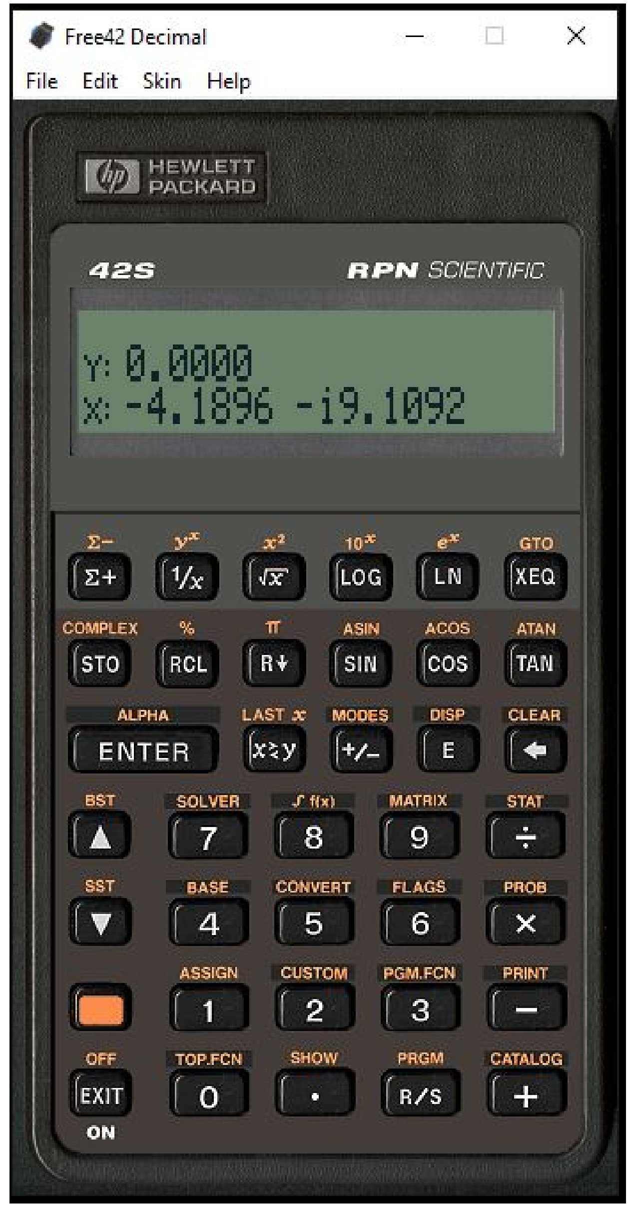
$$= \left(\frac{e^{2}\cos(3)+e^{-3}\cos(2)}{2}\right) + j\left(\frac{e^{2}\sin(3)+e^{-3}\sin(2)}{2}\right)$$

Check in Matlab

```
>> cos(2 + j*3)

ans =  -4.1896 - 9.1092i
```

Check on an HP42

```
2
enter
3
complex
COS

ans = -4.1896 - j9.1092
```



Find y:

$$y = \ln(2 + j3)$$

Express as an exponential

$$2 + j3 = 3.6056\angle 0.9828$$

$$= e^{\ln(3.6056) + j0.9828}$$

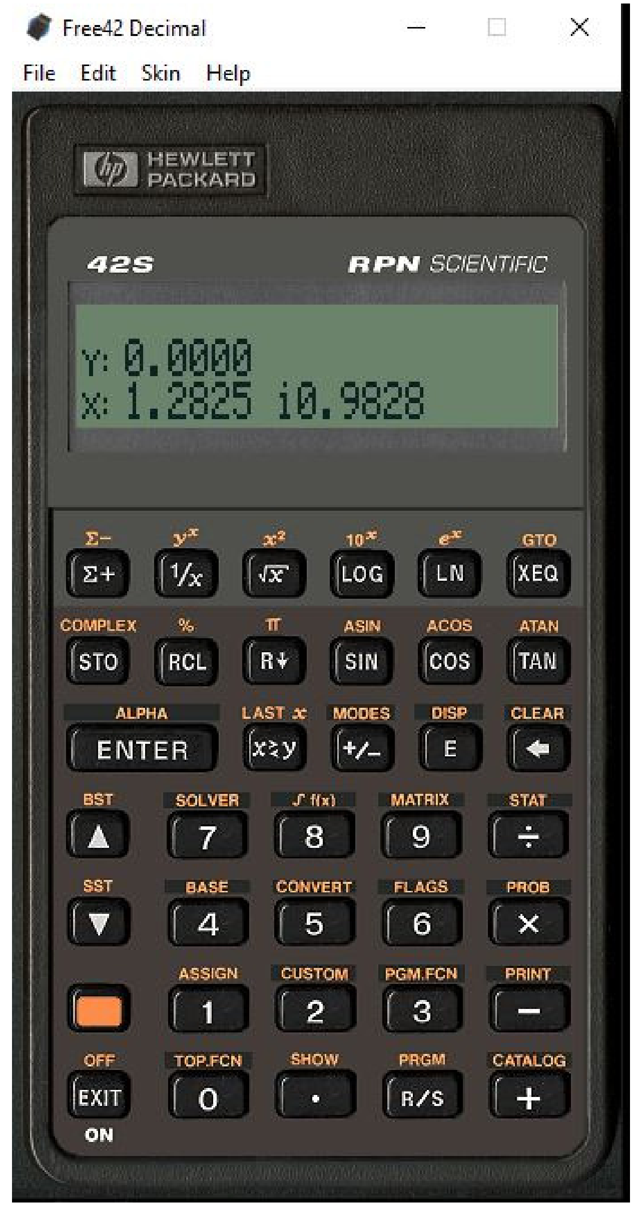
$$\ln(2 + j3) = \ln(3.6056) + j0.9828$$

Check in Matlab

```
>> log(2 + j*3)

ans =   1.2825 + 0.9828i
```

Check with an HP42

```
2
enter
3
ln

ans =   1.2825 + 0.9828i
```



Handout:

- Find y by hand
- Check your answer with Matlab or a calculator

$$y = \sqrt{i}$$

Moral #1: Pretty much anything you do with real numbers you can do with complex numbers. The answer will be complex though.

Moral #2: When dealing with complex numbers, it is a *lot* easier to use Matlab or an HP calculator than doing it by hand...

Summary:

- Complex numbers allow you to represent something with two degrees of freedom
- Addition, subtraction, multiplication, and division all work with complex numbers
- Calculations using complex numbers by hand is really painful.
- Calculations using complex numbers aren't too bad with Matlab or an HP calculator
  - Get familiar with a calculator which does complex numbers before you get to Circuits I
  - You're going to need it for midterms for pretty much all courses in ECE