Math 129: Linear Algebra

ECE 111 Introduction to ECE

Jake Glower - Week #4

Please visit Bison Academy for corresponding lecture notes, homework sets, and solutions

Introduction

Algebra: Solve one equation for one unknown

$$2(x+3)+5x=10x+20$$

Example: Determine R1 as a function of { V0, V1, R2} given

$$V_1 = \left(\frac{R_1}{R_1+R_2}\right) V_0$$

Solution

$$(R_1+R_2)V_1 = R_1 V_0$$

$$R_2 V_1 = R_1(V_0 - V_1)$$

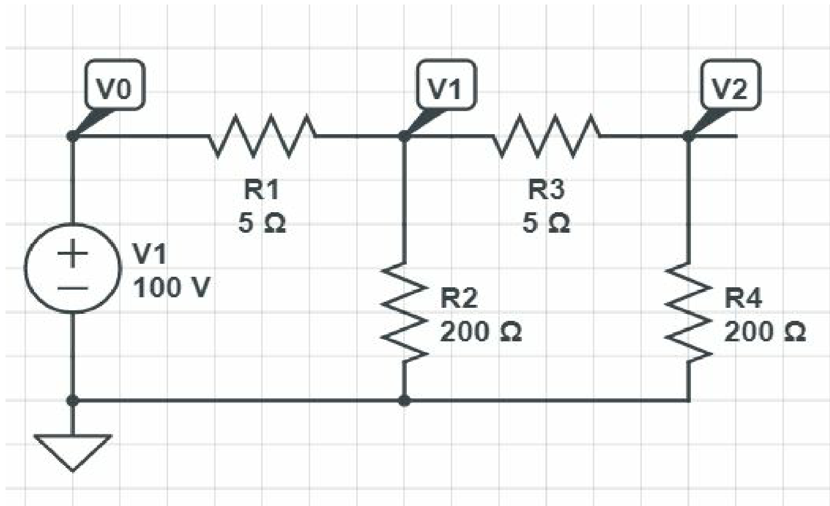
$$\boxed{R_1 = \left(\frac{V_1}{V_0 - V_1}\right) R_2}$$

this is how an ohm meter works

Algebra: Solving 2 equations for 2 unknowns

$$2x + 3y = 10$$

$$5x - 7y = 20$$



Step 1: Solve for x:

$$x = \left(\frac{10-3y}{2}\right)$$

Substitute

$$5\left(\frac{10-3y}{2}\right) - 7y = 20$$

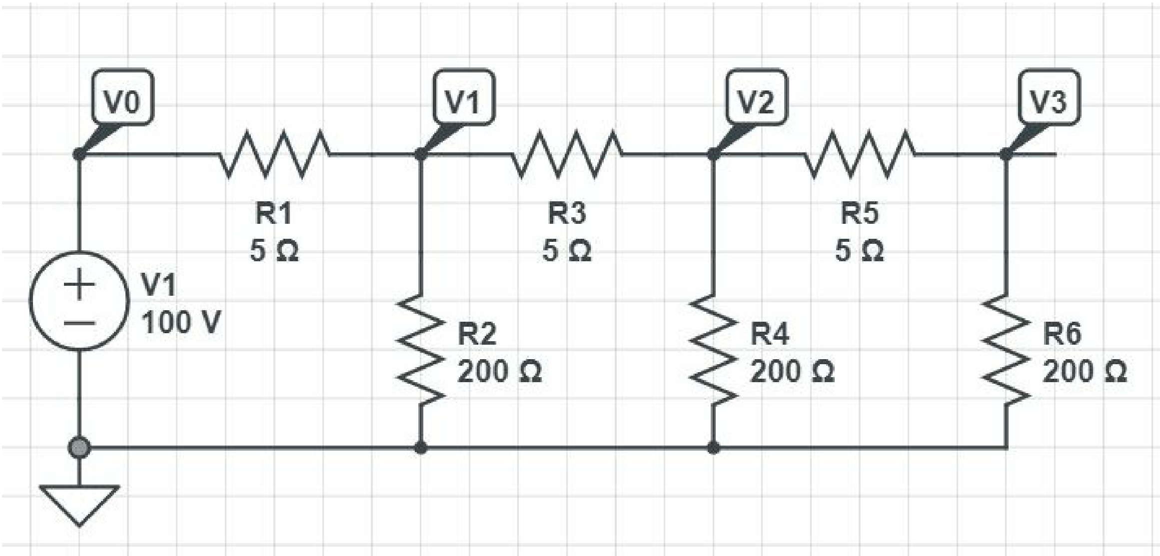
You now have one equation for one unknown

Algebra: Solving 3 equations for 3 unknowns

$$2x + 3y + 4z = 10$$

$$5x - 7y + 2z = 5$$

$$x + y + z = 2$$



Step 1: Solve for x

$$x = \left(\frac{10-3y-4z}{2}\right)$$

Substitute

$$5\left(\frac{10-3y-4z}{2}\right) - 7y + 2z = 5$$

$$\left(\frac{10-3y-4z}{2}\right) + y + z = 2$$

You now have 2 equations and 2 unknowns

- Algebra works, but gets really unwieldy past 2 equations and 2 uknowns
- We need a better tool

Linear Algebra:

- Solve N equations for N unknowns
- Solution uses matrices
- Matlab excels at this type of problem

Example: Solve for { a, b, c }

$$3a + 4b + 5c = 10$$

$$5a + 6b - c = 20$$

$$a + b + c = 2.$$

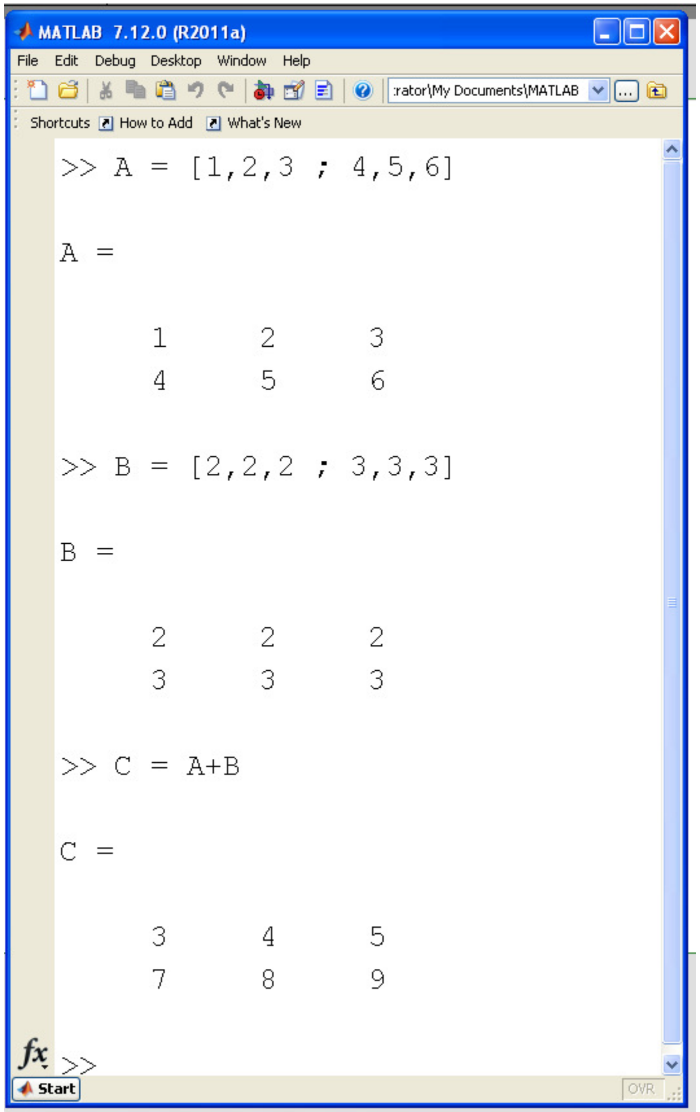
Matrix Definition and Properties.

Dimension: rows x columns

- Example: A is a 2x3 matrix

```
A = [1,2,3 ; 4,5,6]

     1      2      3
     4      5      6
```



Matrix Addition:

- Add each element
- Dimensions must match

Multiplication:

- Inner dimension must match
- $C_{2x1} = A_{2x3} B_{3x1}$

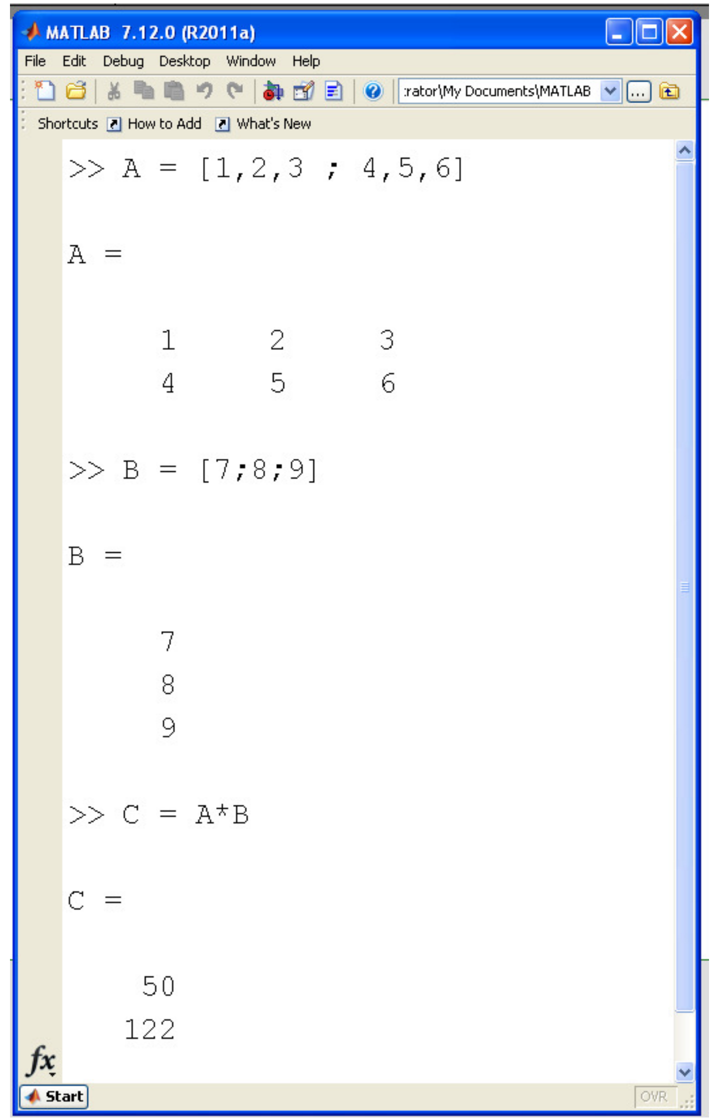
Element i,j of matrix C is computed as

$$c_{ij} = \sum_k a_{ik} b_{kj}$$

Note that matrix multiplication is *not* commutative:

$$AB \neq BA$$

```
 C = B*A
??? Error using ==> mtimes
Inner matrix dimensions must agree.
```



Zero Matrix:

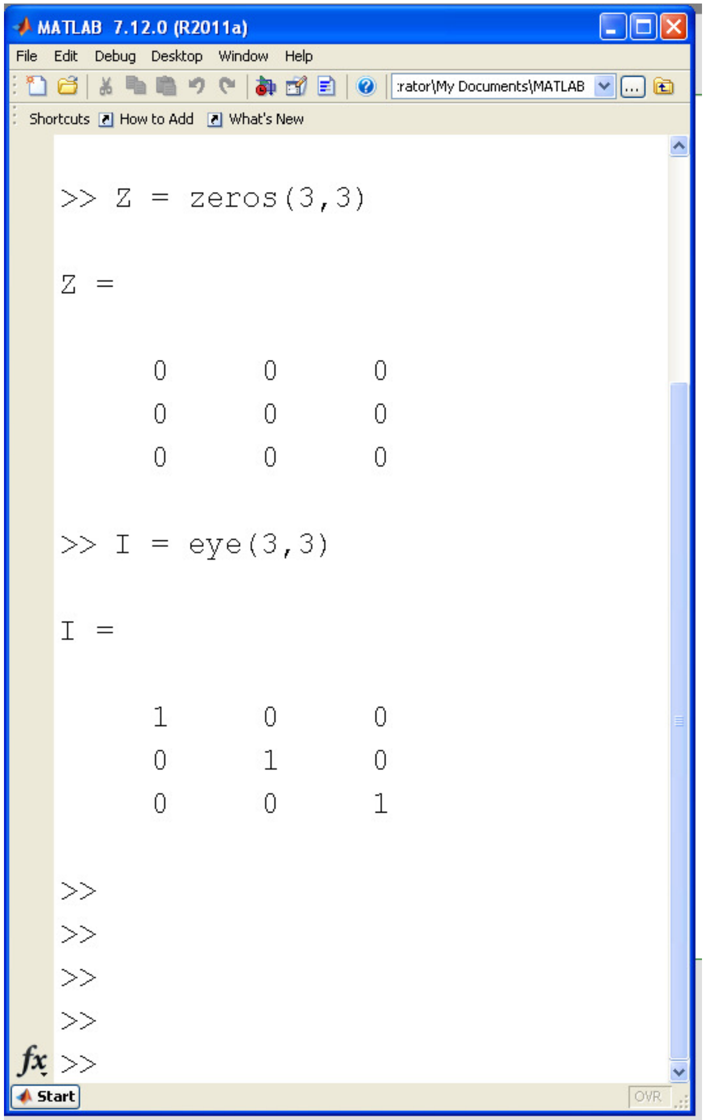
- A zero matrix is a matrix of all zeros.
- The zero matrix behaves like the number zero:
- A + 0 = A
- A * 0 = 0

Identity Matrix:

- NxN matrix
- Diagonal is one
- All other elements are zero
- The identity matrix behaves like the number one:
- A * I = A

Matrix Transpose: A^T

- Swap rows and colums
- A' in Matlab



Matrix Inverse: B is the inverse of A if AB = I

```
A = [1,2,3 ; 4,5, 6; 1 2 1]

    1      2      3
    4      5      6
    1      2      1

B = inv(A)

  -1.1667    0.6667   -0.5000
   0.3333   -0.3333    1.0000
   0.5000         0   -0.5000

A*B

    1      0      0
    0      1      0
    0      0      1
```

Solving N equations for N unknowns

Express in matrix form

$$Y_{Nx1} = B_{NxN} A_{Nx1}$$

where

- A is a matrix of your N unknowns
- B is a basis function and
- Y the result for these N equations

The solution is then

$$A = B^{-1} Y$$

Example: Solve the following set of 3 equations for 3 unknowns:

$$3a + 4b + 5c = 10$$

$$5a + 6b - c = 20$$

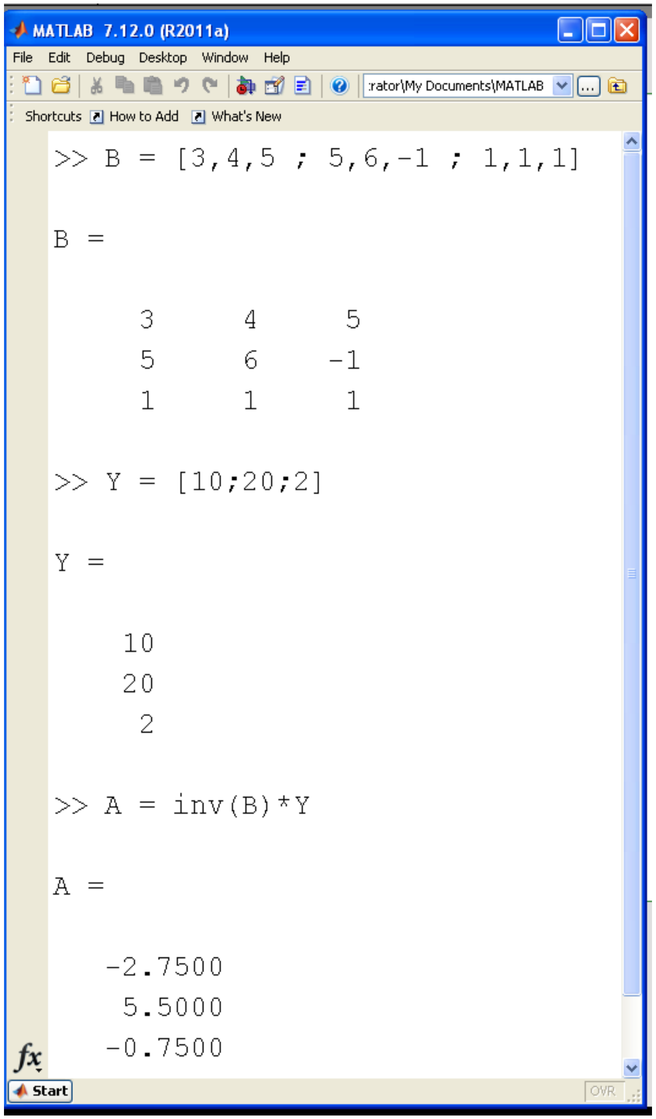
$$a + b + c = 2$$

Step 1: Group terms and write in matrix form:

$$\begin{bmatrix} 3 & 4 & 5 \\ 5 & 6 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 2 \end{bmatrix}$$

Step 2: Invert and solve

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 & 4 & 5 \\ 5 & 6 & -1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 20 \\ 2 \end{bmatrix} = \begin{bmatrix} -2.7500 \\ 5.5000 \\ 0.7500 \end{bmatrix}$$



Example #1

Over the range of (0, 1.5), approximate

$$y = \sin(x) \approx ax + b$$

Solution: With 2 unknowns, we need 2 equations.

- Pick the endpoints

Place in matrix form

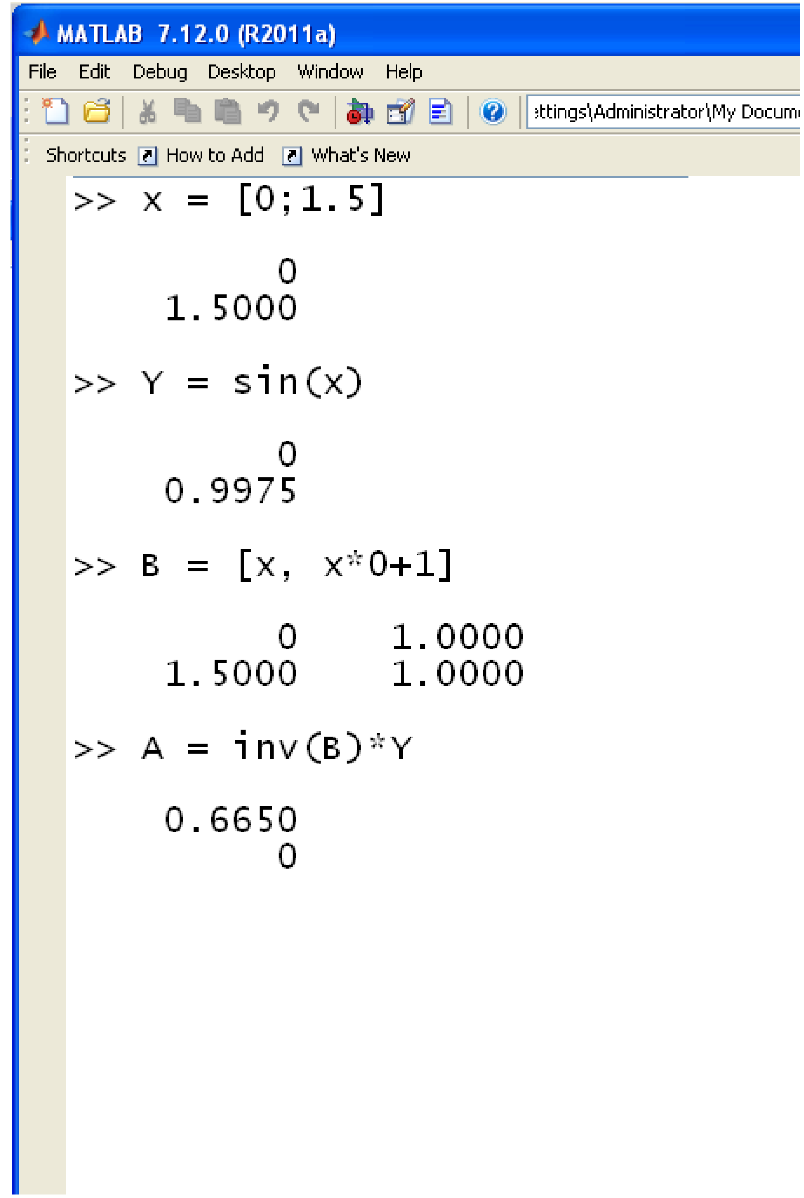
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$Y = BA$$

$$A = B^{-1}A$$

Result:

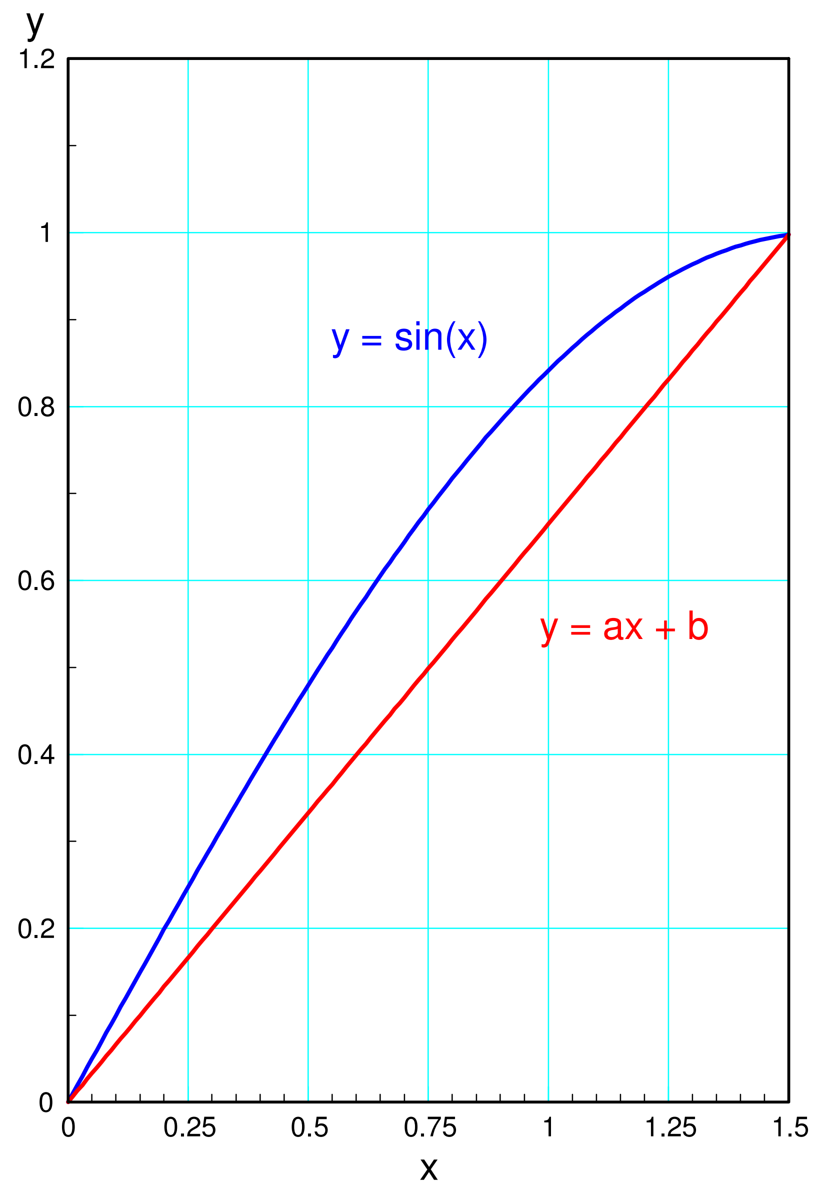
$$\sin(x) \approx 0.6650x + 0$$



Note: This solution defines a line that passes through (x1, y1) and (x2, y2)

(the endpoints)

```
>> x = [0:0.01:1.5]';
>> y = sin(x);
>> B = [x, x*0+1];
>> plot(x,y,'b',x,B*A,'r')
```



Example 2:

Approximate sin(x) with a parabola

$$y = \sin(x) \approx ax^2 + bx + c$$

Solution:

- There are three unknowns
- Create 3 equations for 3 unknowns
- Pick 3 points (x1, x2, x3)

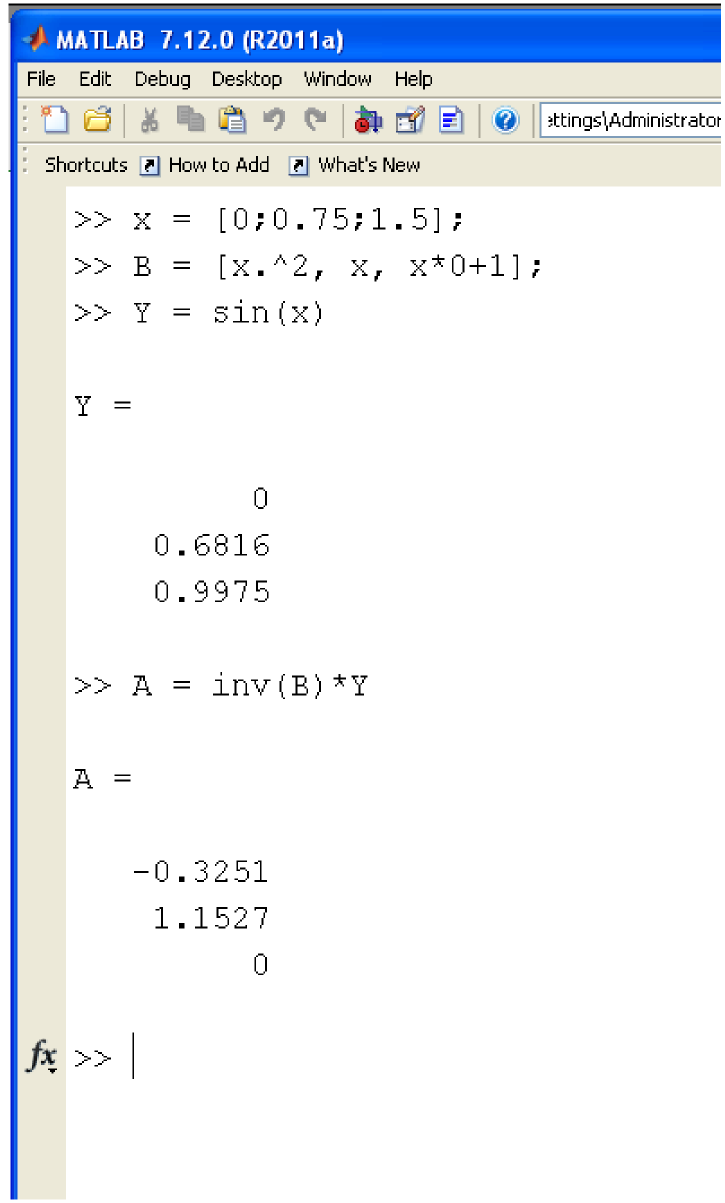
$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$Y = BA$$

$$A = B^{-1} Y$$

result:

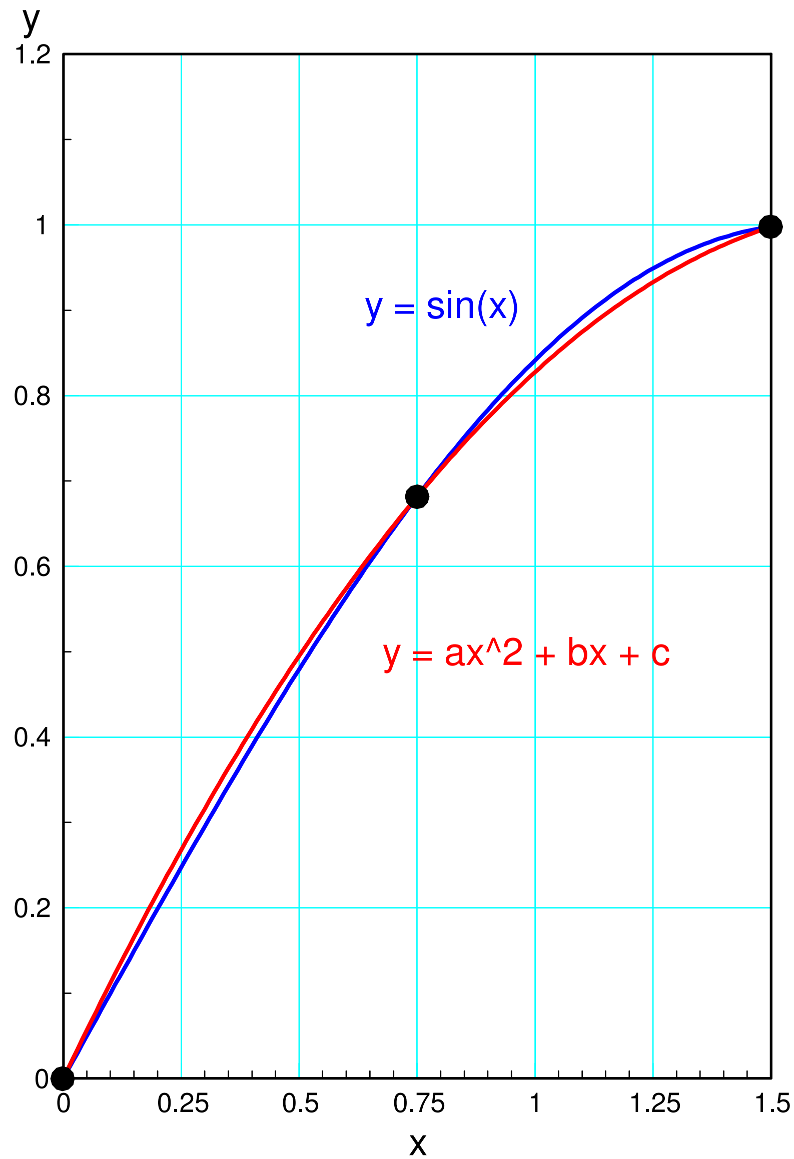
$$\sin(x) \approx -0.3251x^2 + 1.1527x + 0$$



Note: This solution defines a parabola that passes through

- (x1, y1),
- (x2, y2),
- (x3, y3)

```
>> x = [0:0.01:1.5]';
>> y = sin(x);
>> B = [x.^2, x, x.^0];
>> plot(x,y,'b',x,B*A,'r')
```



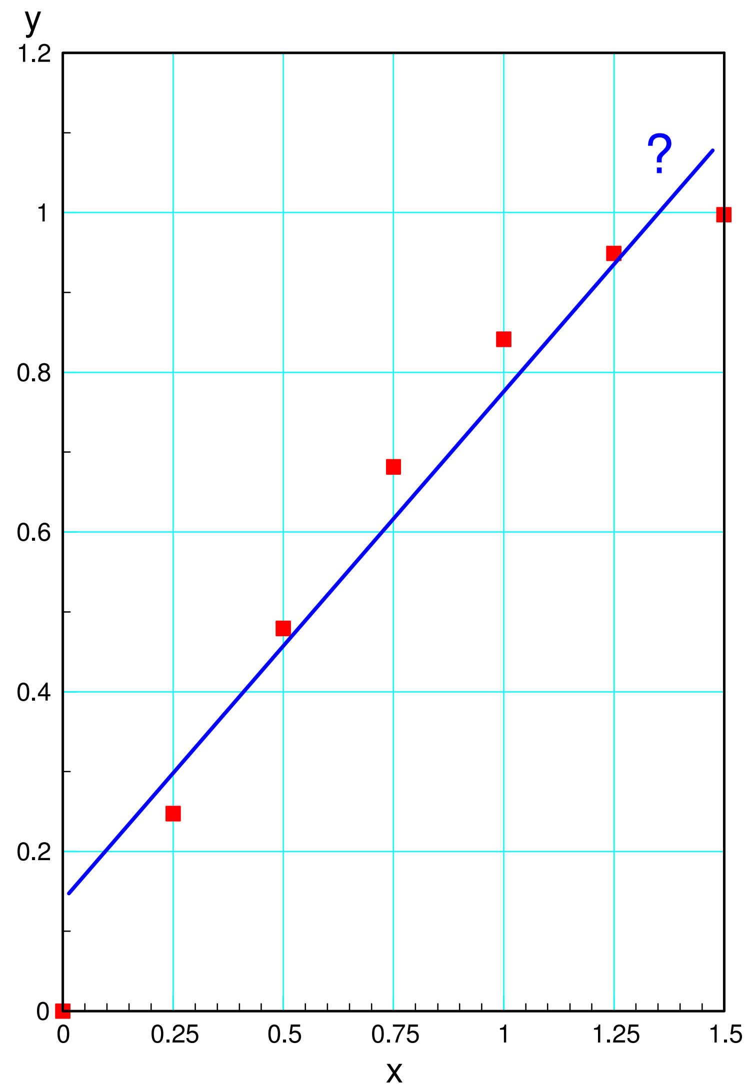
What happens if you have more equations than unknowns?

Previous solution ignores data outside of points chosen

- 2 points for y = ax + b
- 3 points for $y = ax^2 + bx + x$

How do you include all of the data in the calculations?

What is the "best" approximation?



Least Squares Solution

Define "best" to be the curve that minimizes the sum squared difference

- a.k.a. *least squares*

Solution: Assume you have N equations for M unknowns

$$Y_{nx1} = B_{nxm} \cdot A_{mx1}$$

B is not invertable, so multiply on the left by B^T

$$B^T_{mxn} \cdot Y_{nx1} = B^T_{mxn} \cdot B_{nxm} \cdot A_{mx1}$$

Multiply on the left by $(B^T B)^{-1}$

$$(B^T B)^{-1} B^T Y = A$$

This is the least-squares curve fit

Example 3:

Use seven points to approximate

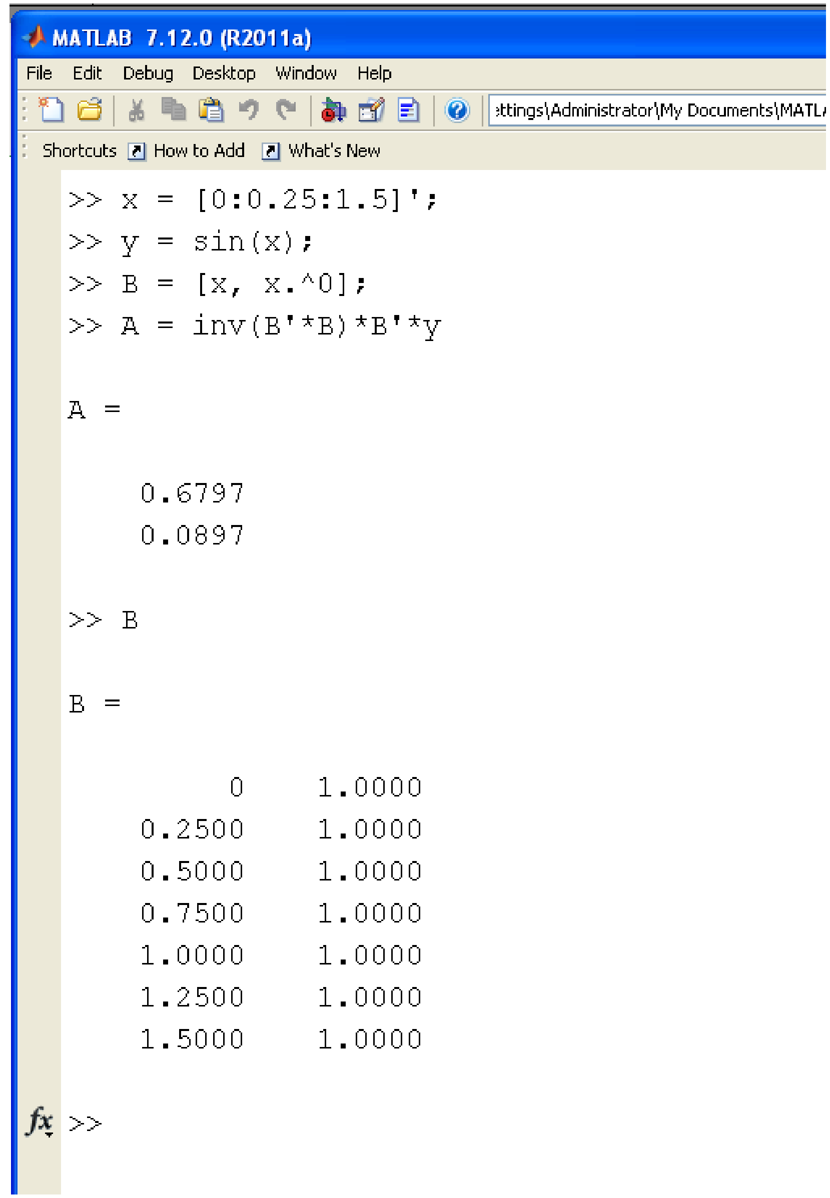
$$y = \sin(x) \approx ax + b$$

Define the basis matrix, B, to be

$$B = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \end{bmatrix}$$

This results in

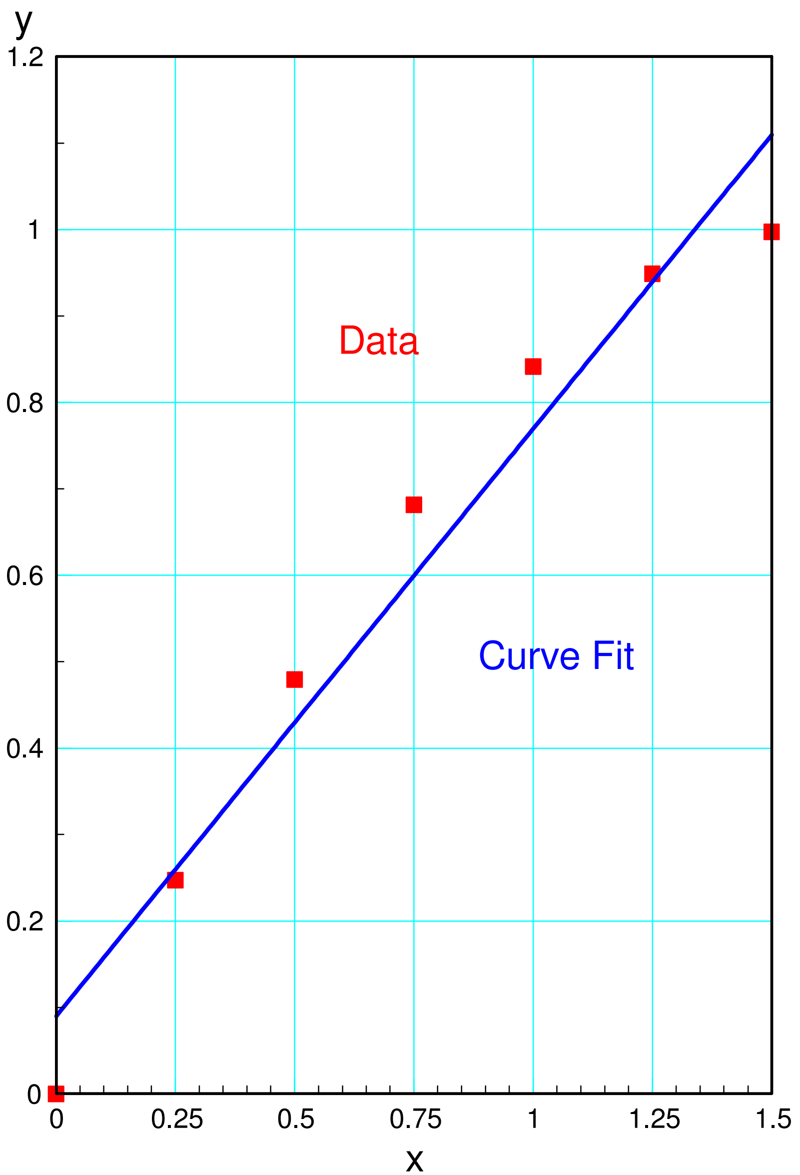
$$\sin(x) \approx 0.6796x + 0.0897$$



This line minimizes the sum squared difference between

- your data and
- the curve fit (the line)

```
>> x0 = [0:0.01:1.5]';
>> B = [x0, x0.^0]
>> plot(x,y,'r+',x0,B*A,'b')
```



Example 4:

Use seven points to approximate

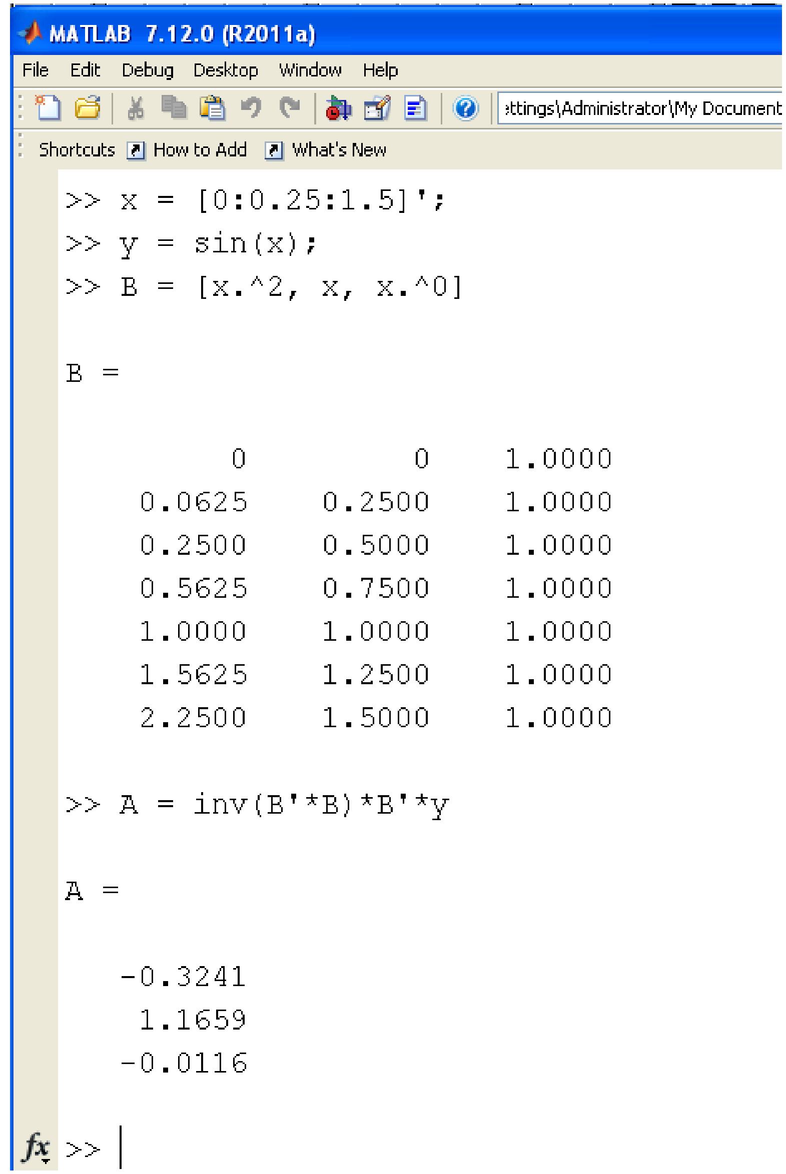
$$y = \sin(x) \approx ax^2 + bx + c$$

Define the basis matrix, B, to be

$$B = \begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ \vdots & \vdots & \vdots \end{bmatrix}$$

This results in

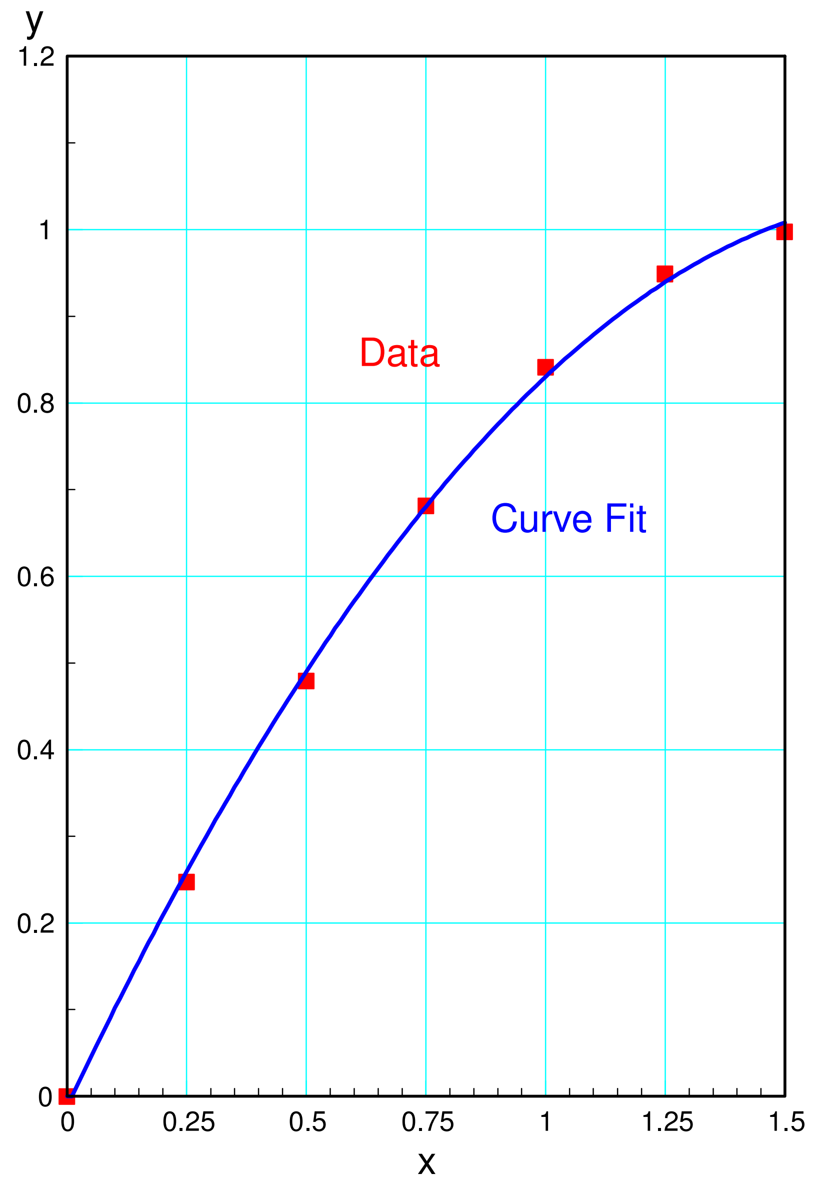
$$\sin(x) \approx -0.3241x^2 + 1.1659x - 0.0116$$



This line minimizes the sum squared difference between

- your data and
- the curve fit (the line)

```
>> x0 = [0:0.01:1.5]';
>> B = [x0.^2, x0, x0.^0]
>> plot(x,y,'r+',x0,B*A,'b')
```



Fun with Curve Fitting

With least squares, you can curve fit anything

- including real data

Let's curve fit

- Artic sea ice cover
- Fargo's temperature
- Global CO2 levels
- Global temperatures

and see what the data tells us....

Arctic Ice Levels

- National Sea and Ice Data Center
- http://nsidc.org/arcticseaicenews/charctic-interactive-sea-ice-graph/

The area covered by sea ice in the Arctic has been measured by the National Sea and Ice Data Center since 1979.

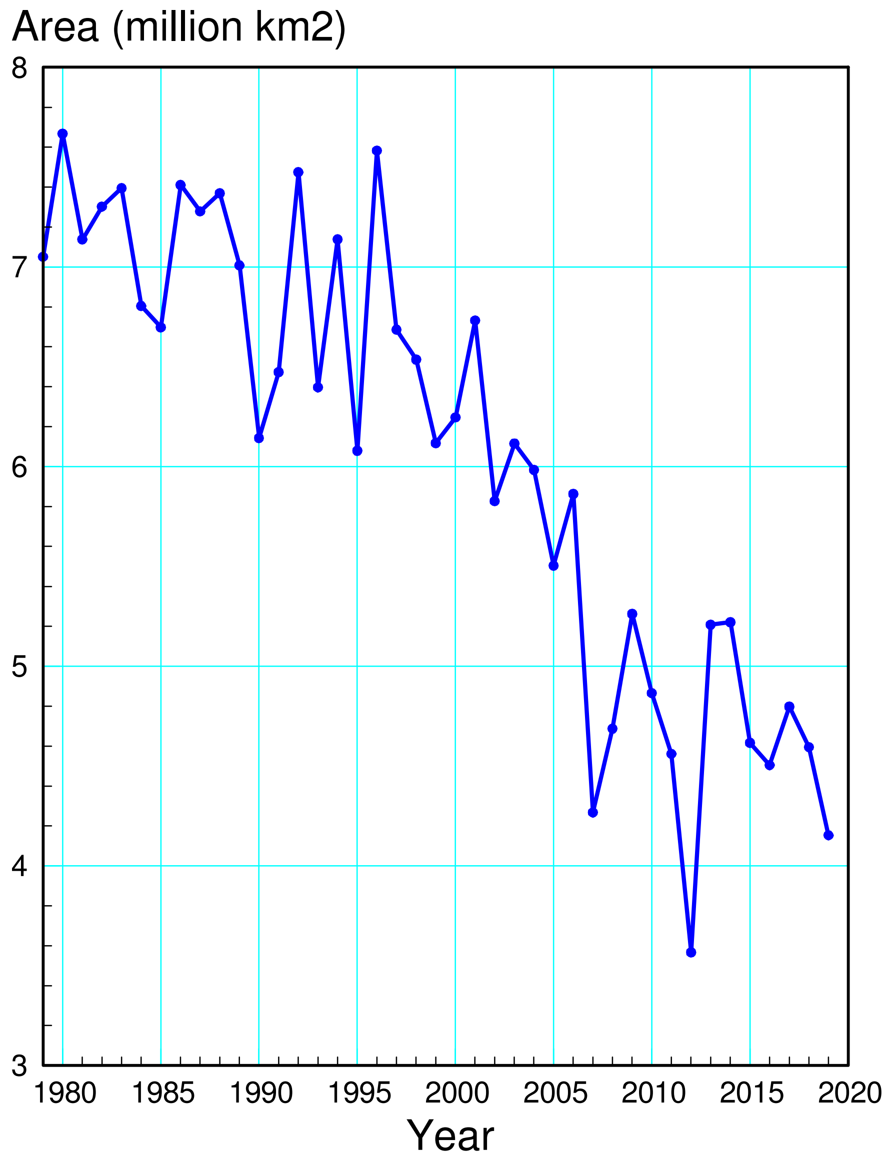
- Record the minimum ice level each year
- Find a linear curve fit for this data
- Determine when the Arcic will be ice free

41 data points

- 41 equations

2 unknowns

- y = ax + b



Least Squares Solution

Step 1: Paste the data into Matlab

```
DATA = [  <paste > ];
year = DATA(:,1);
ice = DATA(:,2);
```

Solve using least squares

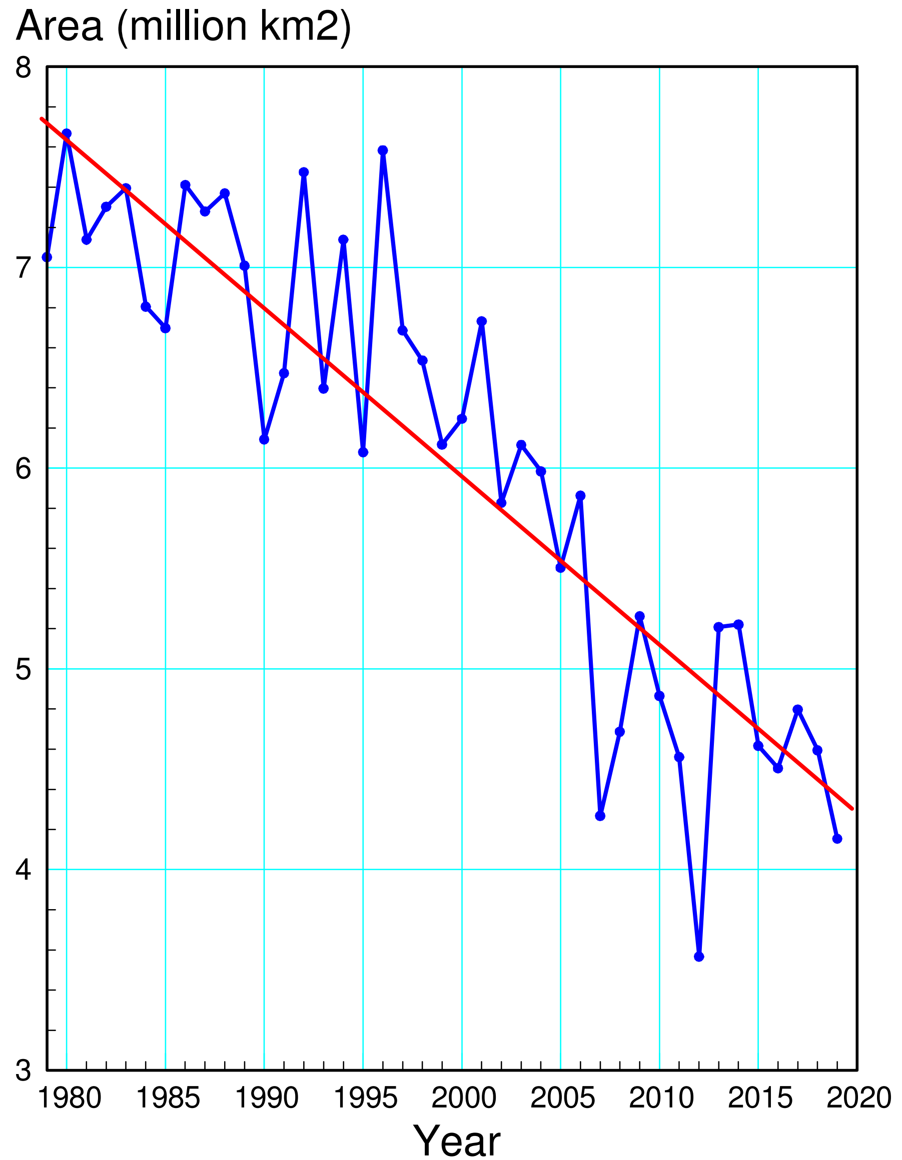
```
B = [year, year.^0];
Y = [ice];

A = inv(B'*B)*B'*Y

 - 0.0844726
   174.68702
```

$$Area \approx -0.0844 \cdot year + 174.68$$

```
plot(y,a,'b.-',y,X*A,'r')
```



Data Analysis

When will the Arctic be ice free?

- First time in 5 million years
- Find the zero crossing

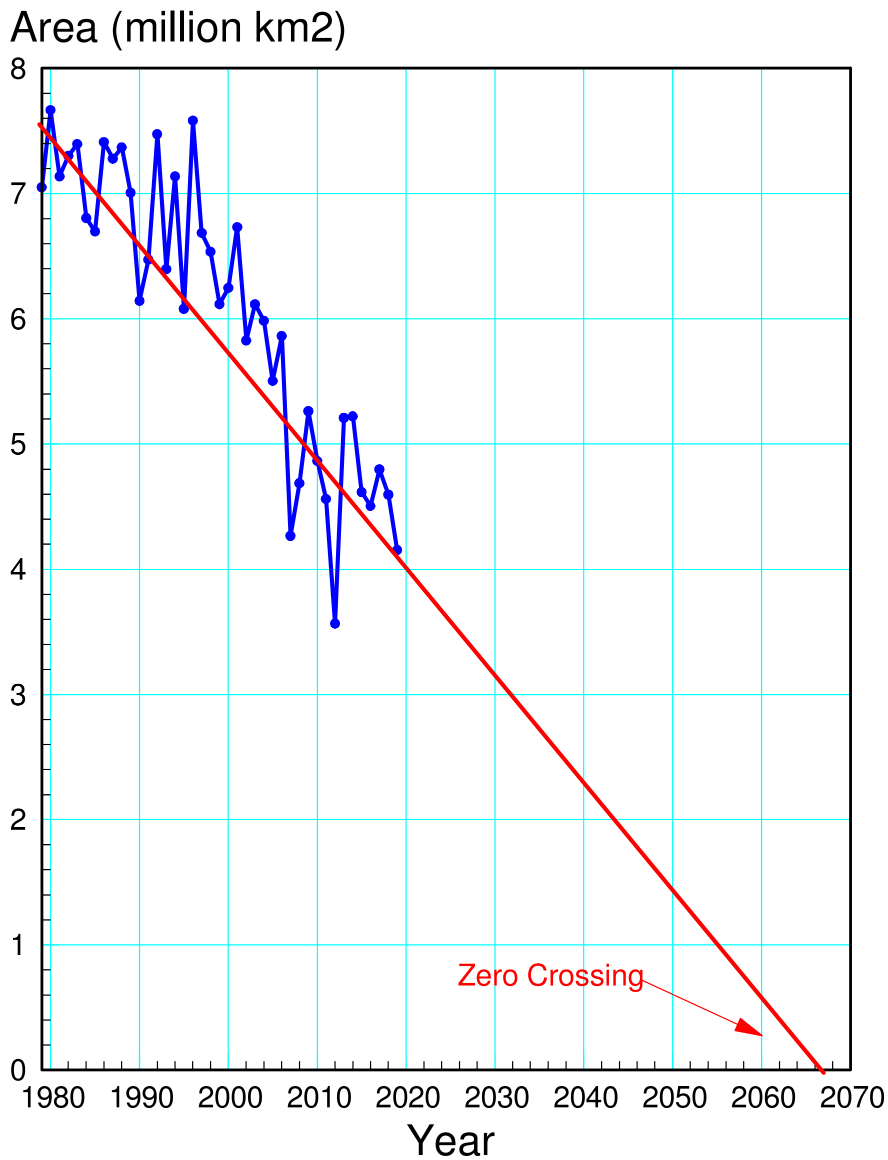
$$Area \approx 0 = -0.0844 \cdot year + 174.68$$

$$year = \left(\frac{174.68}{0.0844}\right) = 2067.97$$

roots() also works

```
roots(A)
    2067.9729
```

Using a linear curve fit, the data predicts that the Arctic will be ice free for the first time in 5 million years in the year 2067.



Fargo Temperatures

Source: Hector Airport

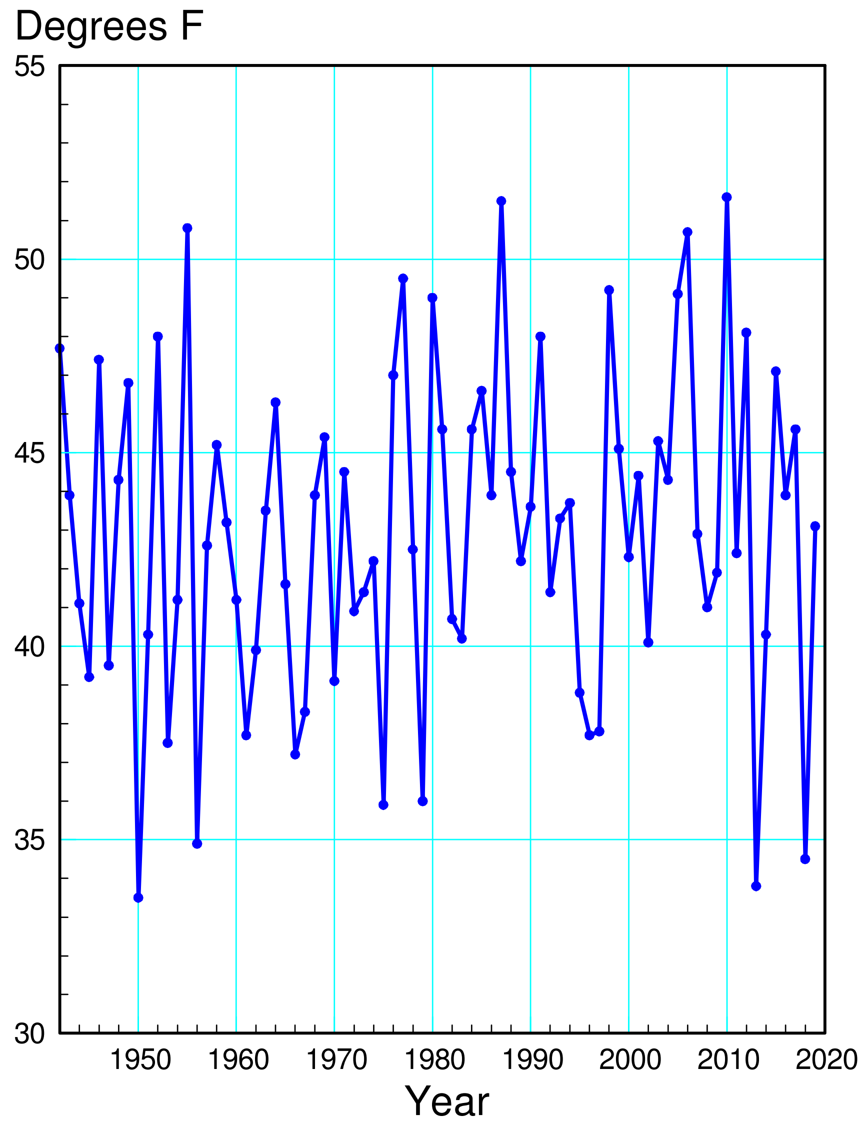
- Mean Temperature in April
- Is there a trend?

Express this in the form of

$$F = ay + b$$

where

- F is the mean temperature and
- y is the year.



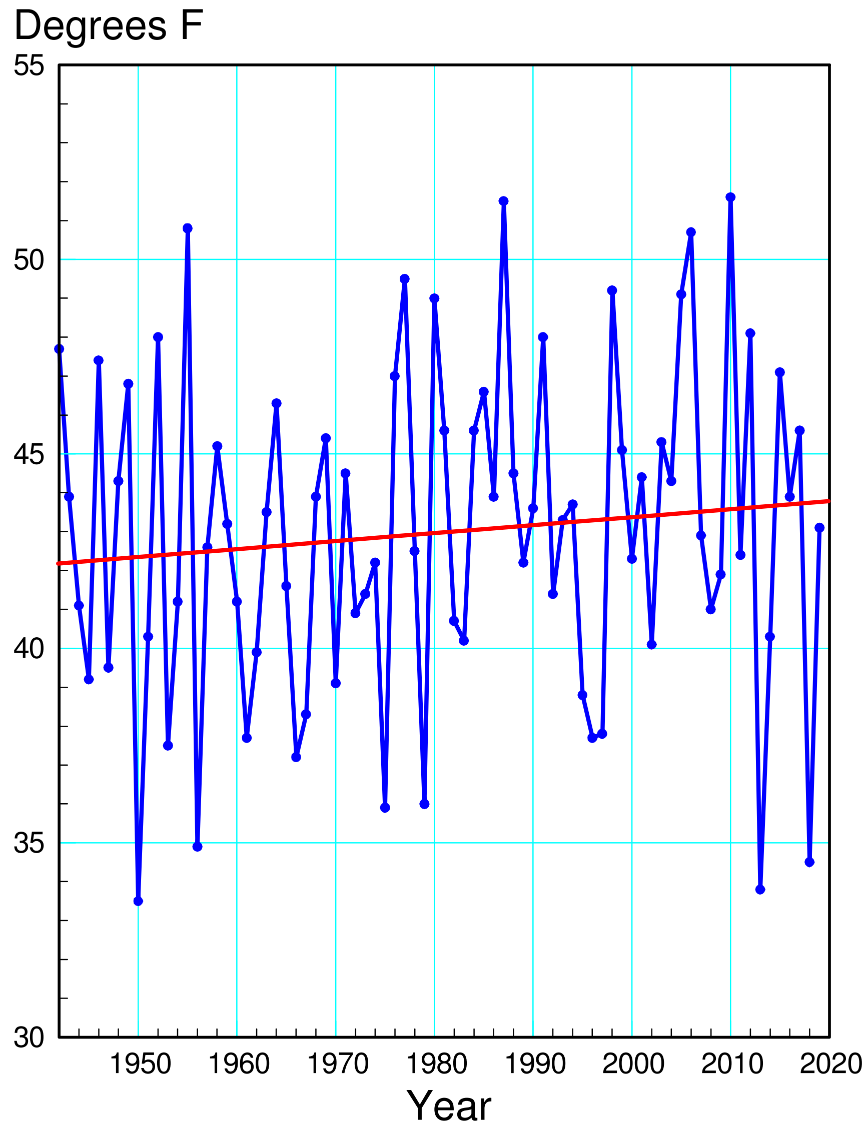
In Matlab:

```
DATA = [
    control V (paste the data)
    ];
 y = DATA(:,1);
 F = DATA(:,5);
 plot(y,F,'.-')

 B = [y, y.^0];
 A = inv(B'*B)*B'*F

    0.0297
  -15.7381

 plot(y,F,'.-',y,B*A,'r')
```

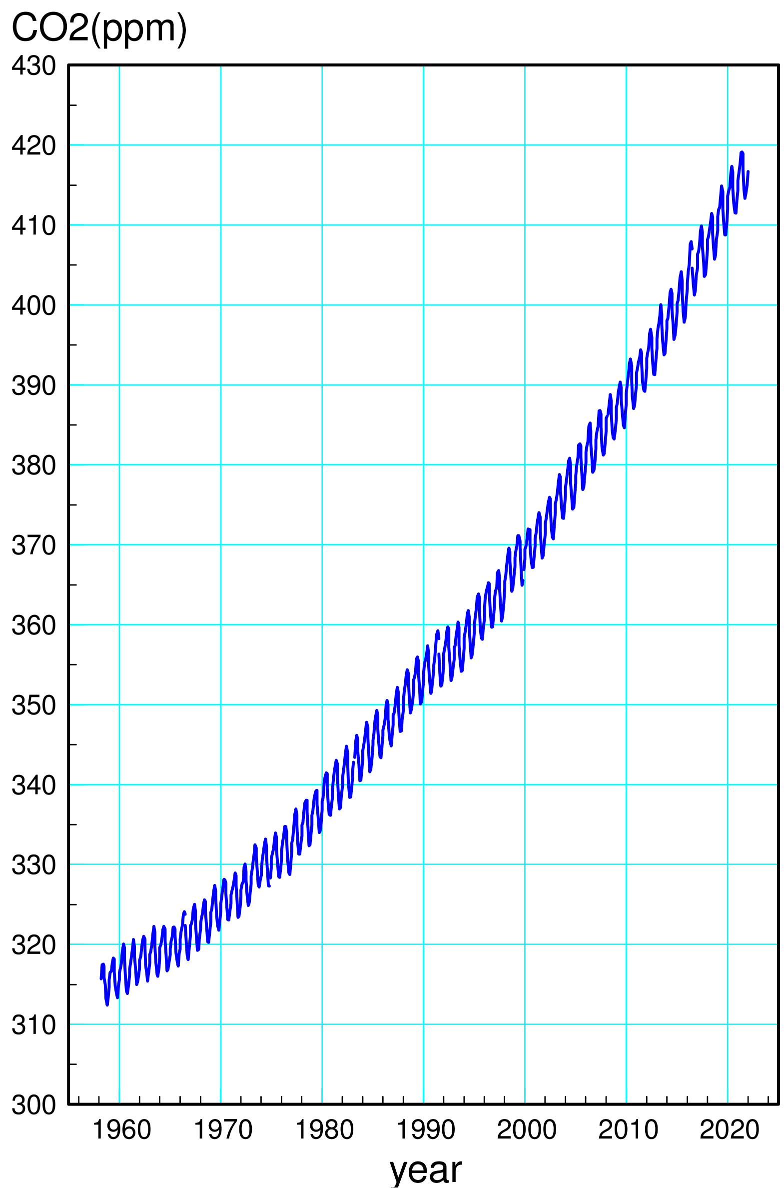


Meaning

- Fargo is warming 0.0297F per year
- +2.37F over 80 years

Atmospheric CO_2 Levels

- Source: NOAA Mauna Loa Observatory
- https://www.esrl.noaa.gov/gmd/ccgg/trends/full.html
- Measured since 1959



Determine a parabolic curve fit

Estimate when CO_2 levels will reach 2000ppm

- Same as what triggered the Permian extinction
- 251 million years ago
- Nearly wiped out all life

Least Squares Curve Fit

Use a parabolic curve fit:

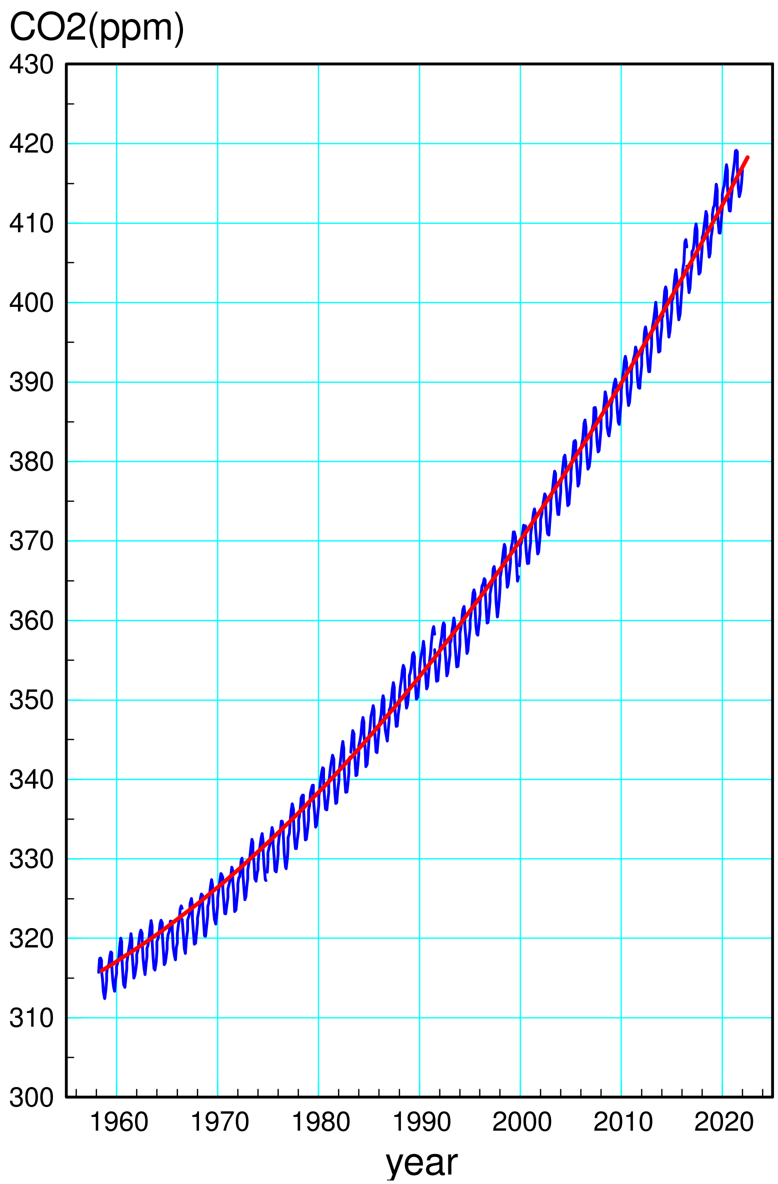
$$CO2 = ay^2 + by + c$$

```
DATA = [
   paste in the data you just copied
   ];

y = DATA(:,3);
CO2 = DATA(:,5);
B = [y.^2, y, y.^0];
A = inv(B'*B)*B'*CO2

 1.3072e-002
-5.0428e+001
 4.8937e+004

plot(y,CO2,'b.-',y,B*A,'r')
xlabel('Year');
ylabel('CO2 ppm');
```



Data Analysis

When will CO2 levels reach 2000 ppm?

$$ay^2 + by + c = 2000$$

Rewrite as

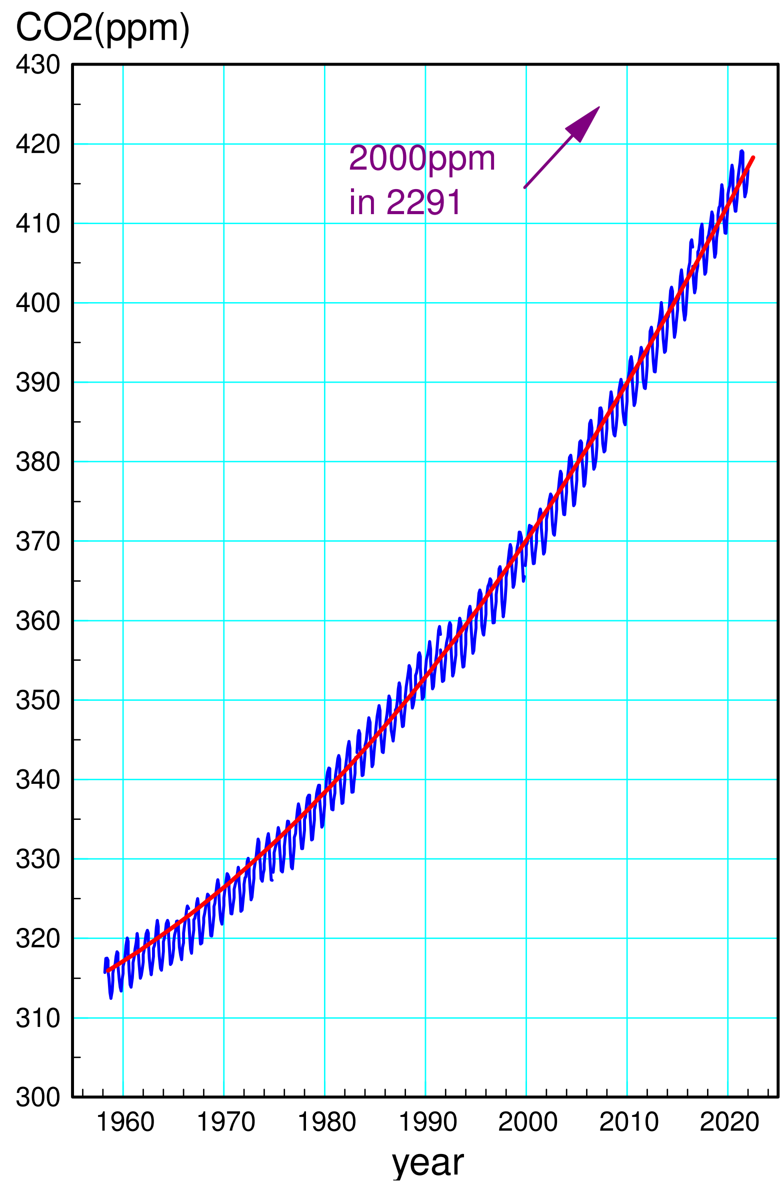
$$ay^2 + by + c - 2000 = 0$$

$$roots\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 2000 \end{bmatrix}\right)$$

```
roots(A - [0;0;2000])
```

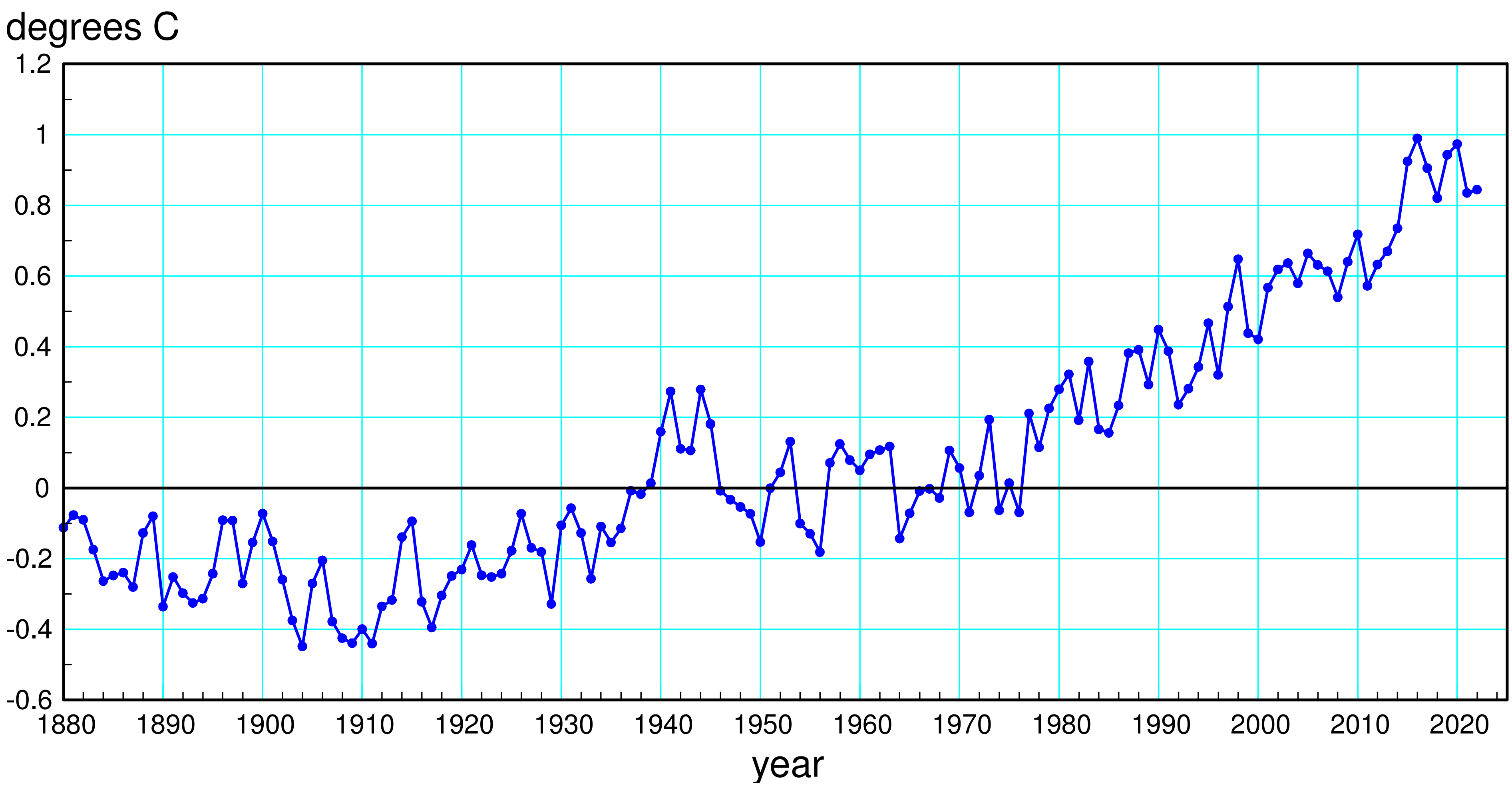
2291.9
1564.3

If nothing changes, we should hit 2000ppm of CO2 in the year 2291.



Global Temperatures

- National Oceanic and Atmosperic Administration
- https://www.ncdc.noaa.gov/cag/global/time-series/globe/land_ocean/p12/12/1880-2022.csv



Global Temperatures (cont'd)

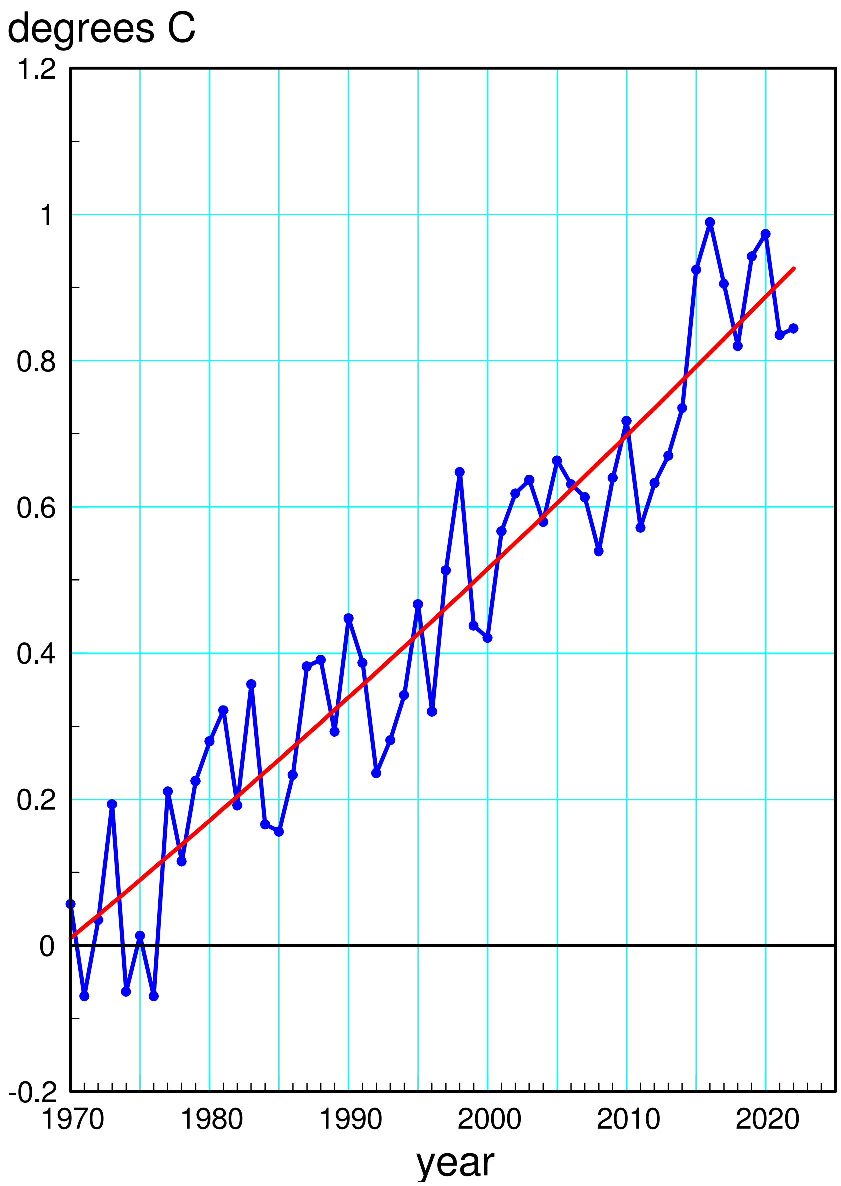
Parabolic curve fit for 1970 .. 2022

```
DATA = [ <paste data 1970..2022> ];
year = DATA(:,1);
dT = DATA(:,2);

B = [year.^2, year, year.^0];
A = inv(B'*B)*B'*dT

   3.5840e-005
  -1.2545e-001
   1.0805e+002

plot(year,dT,'b',year,B*A,'r');
```



Global dT: Data Analysis

When will we reach +10 degrees C?

- The same temperature that triggered the Permian extinction

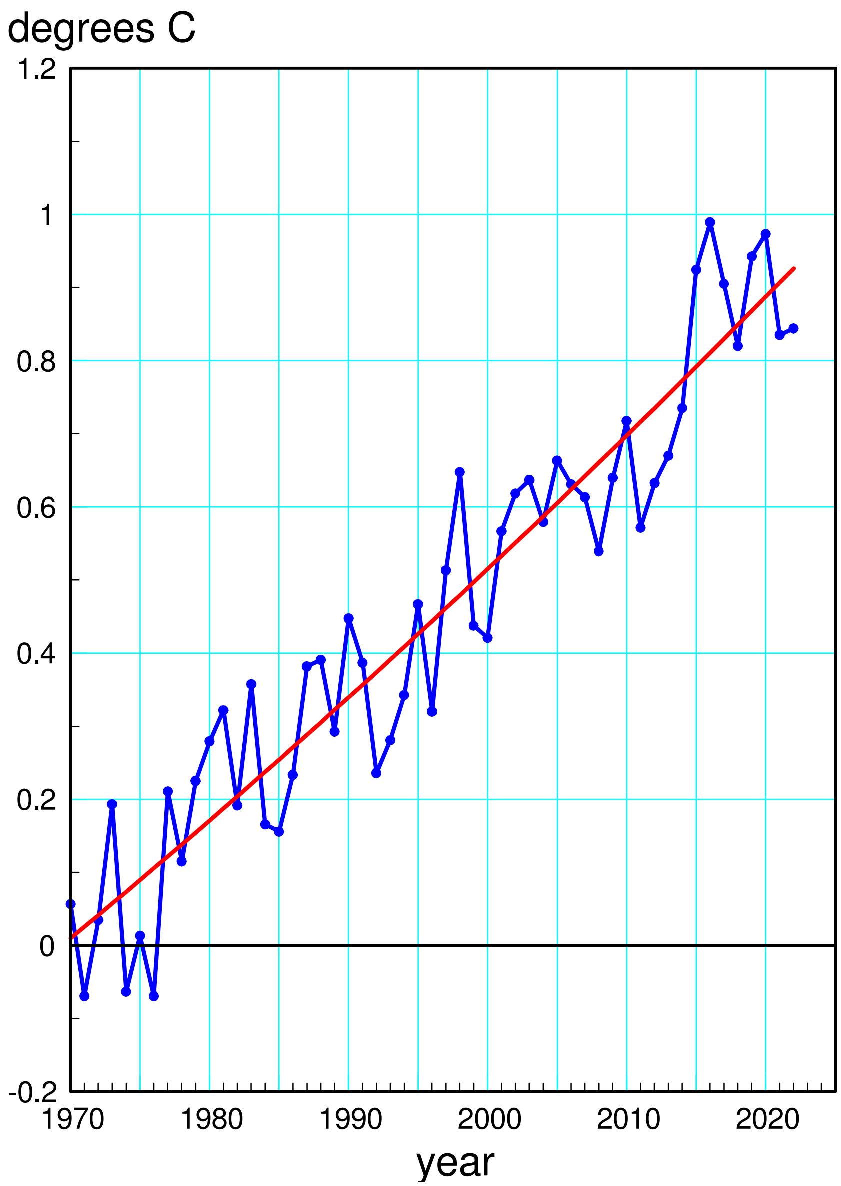
```
>> roots(A - [0;0;10])

   2322.0
   1178.2
```

If nothing changes, we'll reach +10 degrees C in the year 2322

Is this a problem? In 300 years or less...

- The Arctic will be ice free
- CO2 levels will reach 2000ppm
- Global temperatures will reach +10C



The Permian Extinction

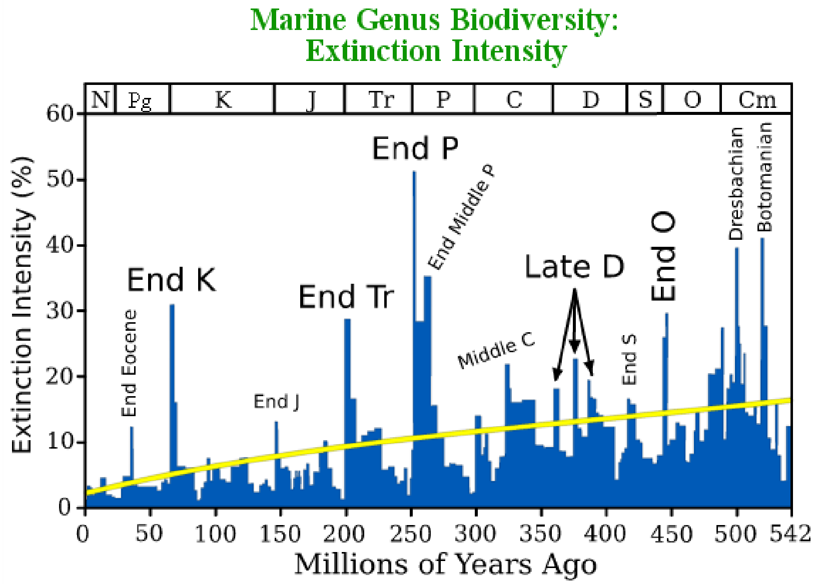
www.Wikipedia.com

Earth has suffered five mass extinction events

- Ordovician–Silurian: 450–440 MYA
- Late Devonian: 375–360 MYA.
- Permian–Triassic: 252 MYA
- Triassic–Jurassic: 201.3 MYA
- Cretaceous–Paleogene: 65MYA

The End-Permian was the largest

- 57% of all families
- 83% of all genera and
- 90% to 96% of all species



What Caused the Permian Extinction?

When Life Nearly Died: The Greatest Mass Extinction of All Time, 2005, by Michael Benton

Step 1: Siberian Trapps

- Massive volcanic erruption
- Lava flow stretches from the Urals to China
- Released huge amounts of CO_2 and SO_2
- Acid rain spurrs the first wave of extinctions

2nd wave

http://i.pinimg.com/736x/db/cb/93/dbcb937238a3c405f7a7f865c1886bf4.jpg

Lava covers coal fields

- Sets the coal on fire
- Raises CO_2 levels to 2000ppm

3rd Wave:

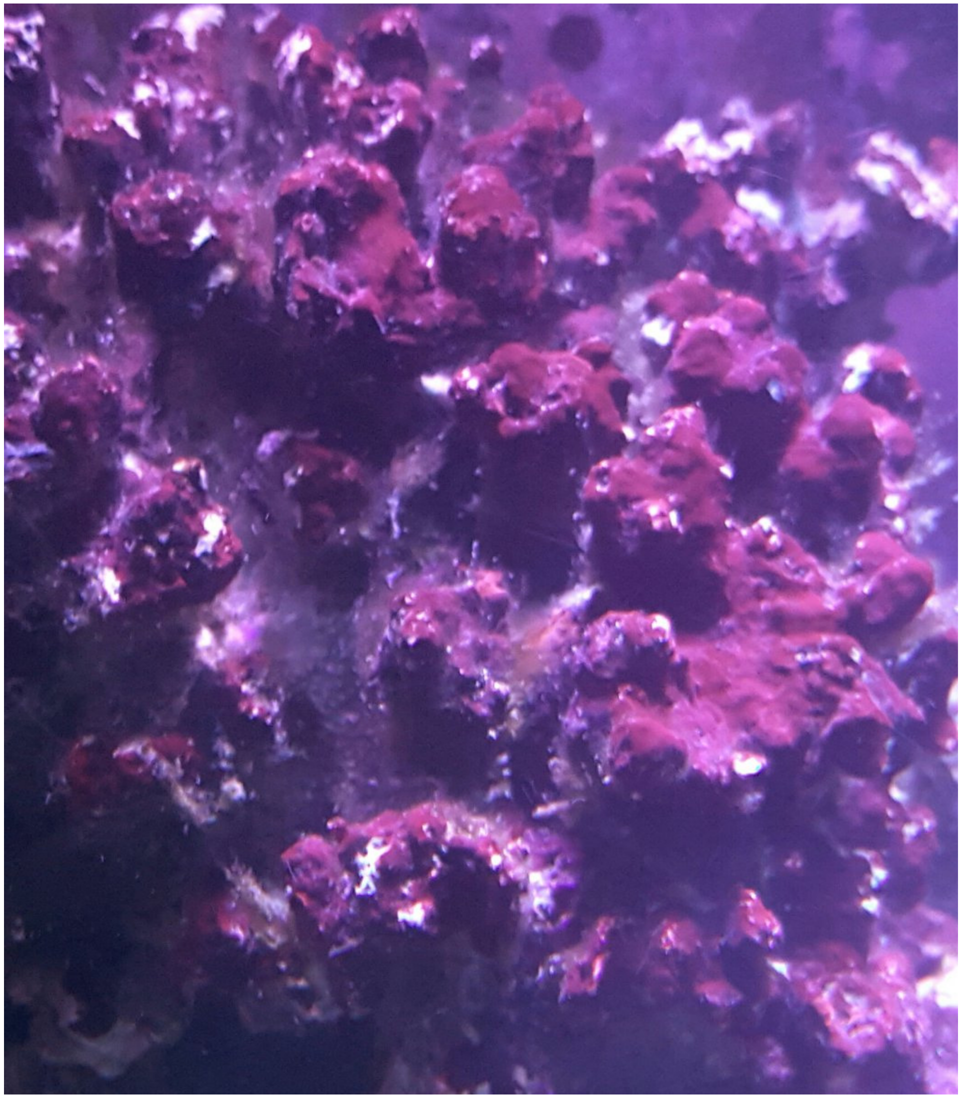
https://geneticliteracyproject.org/wp-content/uploads/2018/10/fire-10-22-18.jpg

- CO_2 raises temperatuers by 10 degrees C
- Triggers another wave of extinctions

4th Wave:

https://www.reef2reef.com/attachments/20160408_211257-1-jpg.352526/

- Warmer temperatures melt the ice caps
- Ocean currents stop
- Without ocean circulation, oxygen levels plummet
- Cyano-bacteria flourish in the oceans
- The air beomes poisoned with cyanide



5th Wave

- Methane hydrates become unstable
- Temperatures rise another 10 degrees C
- 20 degrees C total
- The ocean becomes 130F at the equator

https://i0.wp.com/www.apextribune.com/wp-content/uploads/2014/12/seafloor-methane-released-into-the-pacific-ocean-1024x576.jpg

Net Result

http://english.nigpas.cas.cn/rh/rp/201112/W020111212526403740930.jpg

Life was almost wiped out

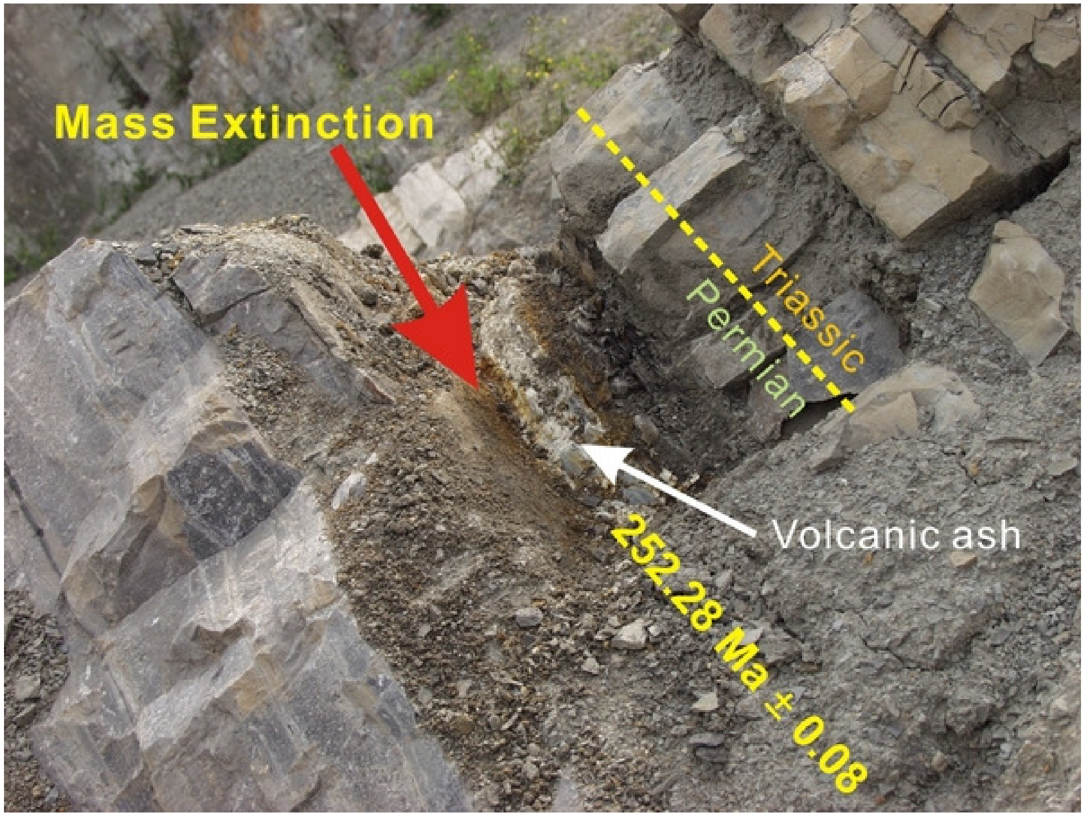
- 57% of all families
- 83% of all genera and
- 90% to 96% of all species

It took almost 10 million years for life to return

- All triggered by +10C temperature rise
- 2000ppm CO2 levels

Is this a repeatable experiment?

- We're going to find out...



Summary:

With matricies, you can solve N equations for N unknowns

$$A = B^{-1} Y$$

- If you can convert a problem to N equations with N unknowns, you can solve
- Very common technique in ECE

If you have more equations than unknowns, you can solve using least-squares

$$A = (B^T B)^{-1} B^T Y$$

- Useful when analyzing actual data (lab results)
- Allows you to see trends in the noise