
Complex Numbers

ECE 111 Introduction to ECE

Jake Glower - Week #12

Please visit [Bison Academy](#) for corresponding lecture notes, homework sets, and solutions



Topics

- Complex Numbers
- Complex Numbers on an HP42 Calculator

Objectives

- Be able to add & subtract complex numbers
 - Be able to multiply and divide complex numbers
 - Be able to switch between rectangular and polar representation for complex numbers
 - Be able to do partial fraction expansion with complex numbers
-

Introduction

Numbers matter.

- Numbers make a difference.
- The existence of mathematical oddities can determine the fate of empires.



The Number Zero:

Why have the number zero?

- Not necessary
- Rome ran a vast empire without the number zero



Roman Numbers:

Romans represented numbers with letters:

- $I = 1$
- $V = 5$
- $X = 10$
- $L = 50$
- $C = 100$
- $IX = 9$ ($10 - 1$)
- $XC = 90$ ($100 - 10$)

Example:

- $27 = XXVII$
 - $109 = CIX$
-

Addition, Subtraction, Multiplication, & Division

Totally possible with Roman numbers

- Rome ran a vast empire
- You know that addition, subtraction, multiplication, and division happened

It's not easy with Roman numbers

$$\begin{array}{r} \text{Addition} \\ \text{XXVII} \\ + \text{CIX} \\ \hline = ? \end{array}$$

$$\begin{array}{r} \text{Multiplication} \\ \text{XXVII} \\ * \text{CIX} \\ \hline = ? \end{array}$$

Arabic Numbers

In contrast, the Arabic number system uses place holders

- Each digit represents an increasing power of ten
- With this number system, you *need* the number zero.

Example:

$$106 = 1 \times 100 + 0 \times 10 + 6 \times 1$$

106 is very different than 16.

With Arabic numbers,

- Addition and subtraction become much easier
 - Multiplication and division remains a challenge
 - But it is far easier than with Roman numbers.
-

Negative Numbers:

Why have negative numbers?

- What does negative one apple mean?
- Why allow this to be represented?

Negative numbers are not necessary

- They do make some calculations easier, however.

Example: Voltage Nodes

- The sum of the current from a node must be zero
 - If you don't mind negative numbers, you don't need to know which direction the current is flowing before you write the equations
-

Accounting & Double-Entry Bookkeeping System

Florence was a world power in the 13th century.

- A remarkable feat considering that Florence is a small city in Italy
- Competing against much larger countries such as Spain, France, and England.



Republic of Florence: a small city state competing for European dominance (Wikipedia)

One reason for this was the invention of the double-entry bookkeeping system.

- Keep track of debits (negative profit)
- Keep track of credits (positive profit)
- Credits minus debits = profit

With this, Florentine merchants could keep track of which ventures were profitable and which were not.

- Everyone else just looked at the return at the end of a venture
 - Ignoring how much money you sunk into it over the course of time
-

"We will bury you"

- Nikita Khrushchev

Accounting can decide the fate of empires

In the 1960's, the Soviet Union *thought* their economy was growing 20% per year

- vs. 4% for the United States

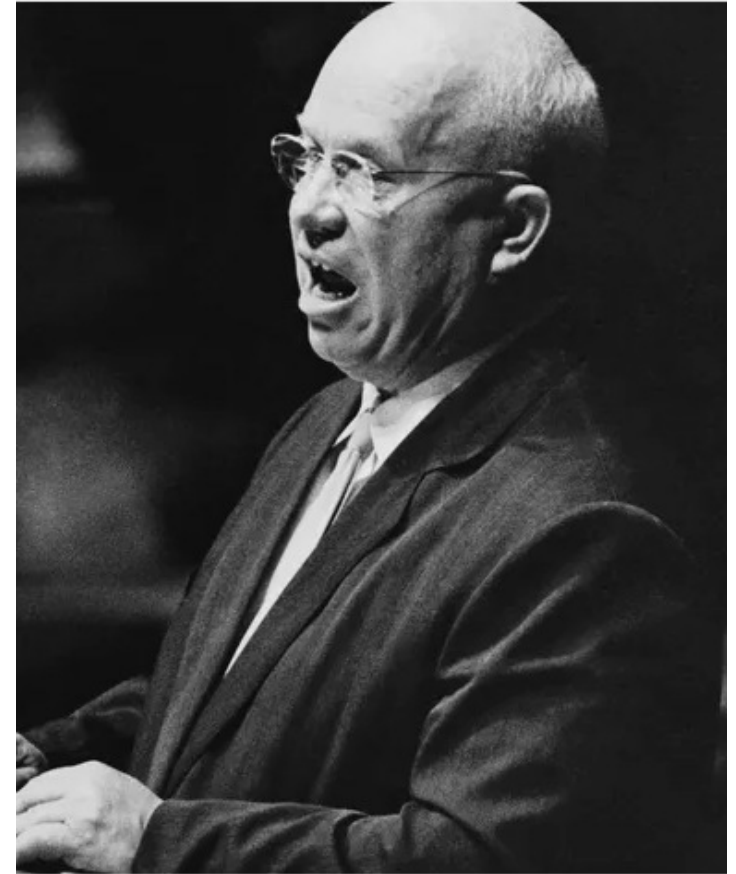
At that rate, by 2000 the Soviet Union would crush the U.S. economically

- Exponential growth is a powerful thing...

Actually, the Soviet economy was shrinking 1-2% per year

Eventually, the size of the government (based upon 20% growth) could not be supported
Accounting & Empires

Accounting can determine the fate of empires



Complex Numbers:

Real numbers work well for DC circuits

- 1st half of EE 206 Circuits I
- Voltages can be expressed by a real number
- Currents can be expressed with real numbers, and
- Resistance's can be expressed with real numbers.

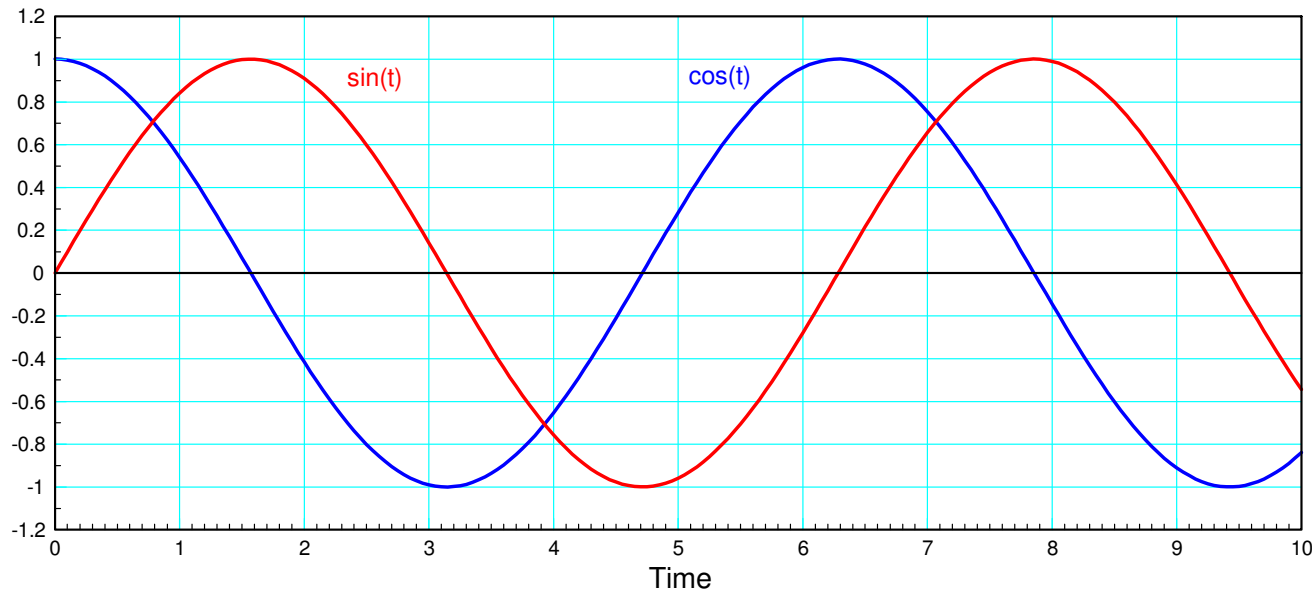
Real numbers have problems when working with AC circuits

- 2nd half of EE 206 Circuits I
 - Rest of ECE curriculum
-

With AC signals, you have two terms:

- $\sin(\omega t)$
- $\cos(\omega t)$

You likewise need two numbers to represent any given voltage or current when dealing with AC signals



$\sin(t)$ and $\cos(t)$. Note that the period is 2π

A generalized sine wave can be represented as

$$x(t) = a \cos(\omega t) + b \sin(\omega t)$$

or

$$x(t) = r \cos(\omega t + \theta)$$

(more on this later).

What this means is, unlike DC, you need three parameters to define a sinusoid:

- The frequency (ω), and
- The cosine and sine terms (a , and b), or
- The amplitude and phase shift (r and θ)

Frequency isn't a problem:

- If you have a linear circuit,
- The frequency doesn't change
- All signals will be the same frequency as the input.

Voltage and current are a problem

- You need a way to represent two parameters:
 - The magnitude of the sine() and cosine() terms in rectangular form, or
 - The amplitude and angle in polar form

Complex numbers provide you with those two degrees of freedom

- Any time you're working with DC signals, real numbers suffice
 - Any time you're working with AC signals, complex numbers are used
-

In short, ECE majors use complex numbers - probably more than any other major.

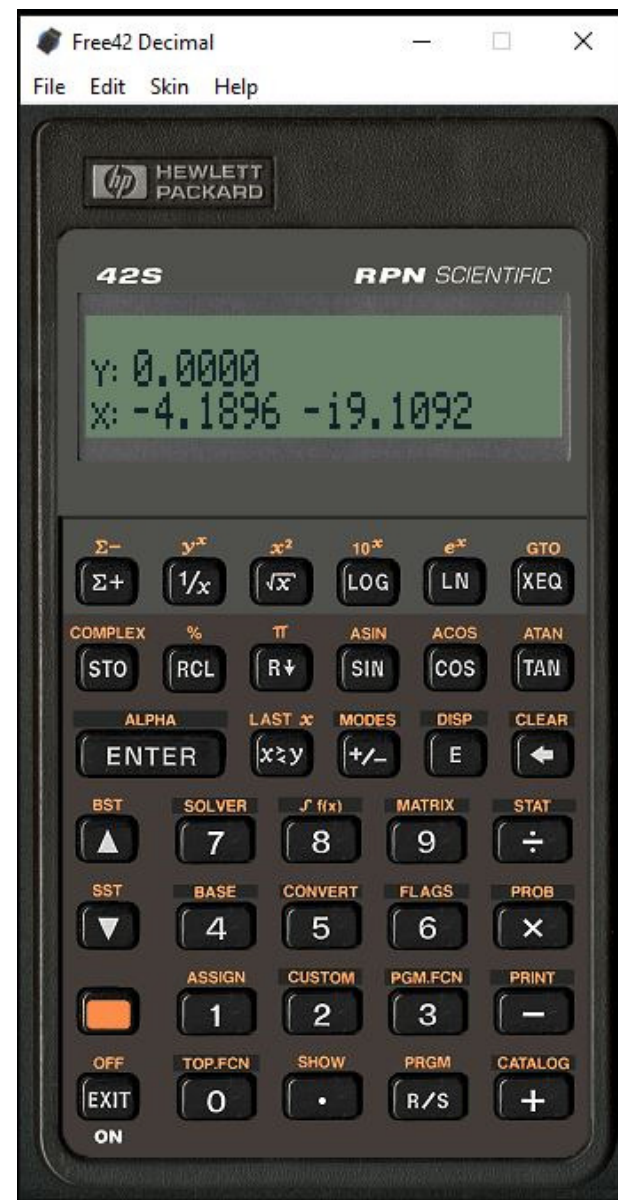
Fortunately, Matlab (and HP calculators) have no problem working with complex numbers.

Free42 is a free app for your cell phone

- It emulates the HP42 calculator
- The *best* calculator ever made for ECE majors
- Deals with complex numbers with ease
- Worth about 10 points on midterms in ECE

It has a learning curve though

- Don't try to learn how to use an HP during a midterm



Complex Numbers

Unlike real numbers, complex numbers have two terms. This allows us to represent the cosine and sine terms for a sinusoid with a single (albeit complex) number.

The basic idea behind complex numbers is to define a term, j , as¹

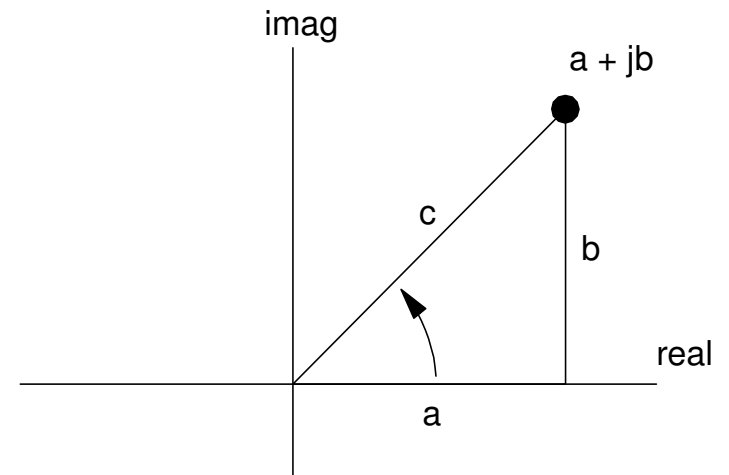
$$j \equiv \sqrt{-1}$$

Any given number can then have a real and a complex part

$$x = a + jb$$

You can express this number in rectangular form ($a + jb$) or polar form

$$x = c \angle \theta$$



¹ Math majors call this term i for imaginary. In ECE, i means current, so we use the letter j to represent the complex part of a number.

Note: The polar form is shorthand notation and actually means

$$c \angle \theta \equiv c \cdot e^{j\theta}$$

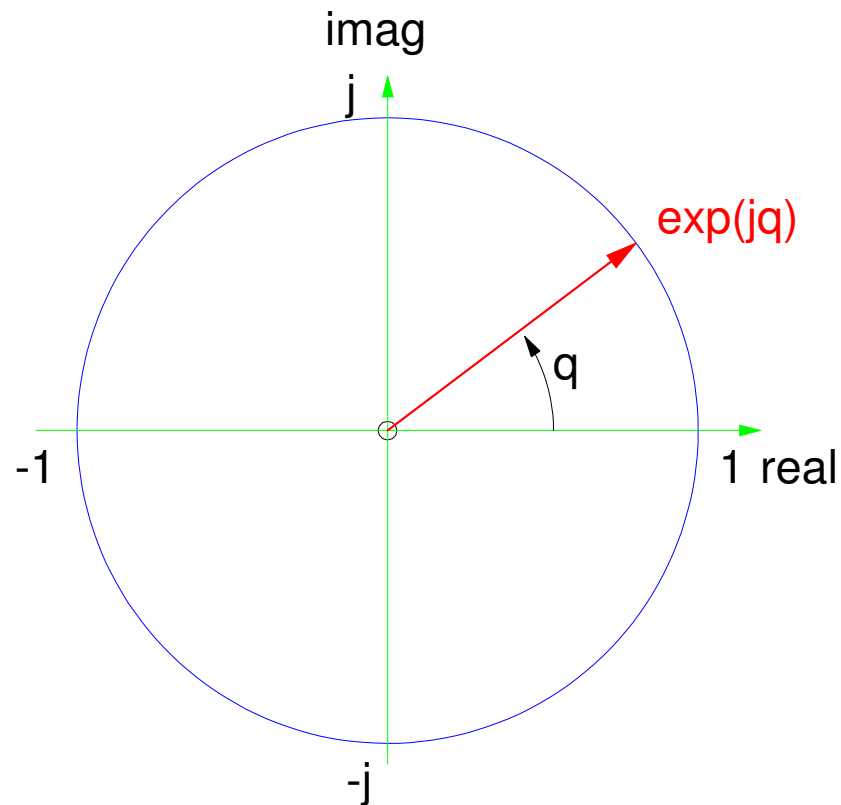
The complex exponential has two terms

$$e^{j\theta} = \cos(\theta) + j \sin(\theta)$$

This leads to Euler's identity

$$\cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$



Proof: Substitute for the complex exponential

$$\left(\frac{e^{j\theta} + e^{-j\theta}}{2}\right) = \frac{1}{2}((\cos \theta + j \sin \theta) + (\cos(-\theta) + j \sin(-\theta)))$$

Using some trig identities

$$\cos(-\theta) = \cos \theta$$

$$\sin(-\theta) = -\sin \theta$$

gives

$$= \left(\frac{1}{2}\right) (\cos \theta + j \sin \theta + \cos \theta - j \sin \theta)$$

$$= \frac{1}{2}(2 \cos \theta)$$

$$= \cos \theta$$

The proof for $\sin(x)$ is similar

Inputting Complex Numbers into Matlab

With Matlab, the default value of j is $\sqrt{-1}$.

If you redefine j , this no longer holds, but you can restore this as

```
>> j = sqrt(-1)
```

To input a number into Matlab in rectangular form, simply use the j variable

```
>> A = 2 + j*3
```

```
A = 2.0000 + 3.0000i
```

You can also input a variable in polar form. $B = 3\angle 1.5$ is input as

```
>> B = 3 * exp(j*1.5)
```

```
B = 0.2122 + 2.9925i
```

(note: Matlab uses radians for its angle units).

The default display in Matlab is rectangular units. To convert to polar, use the `abs` and `angle`

```
>> abs(B)
```

```
ans =      3
```

```
>> angle(B)
```

```
ans =      1.5000
```

Inputting Complex Numbers into an HP42

This uses the complex key. In rectangular mode (yellow - MODES - RECT)

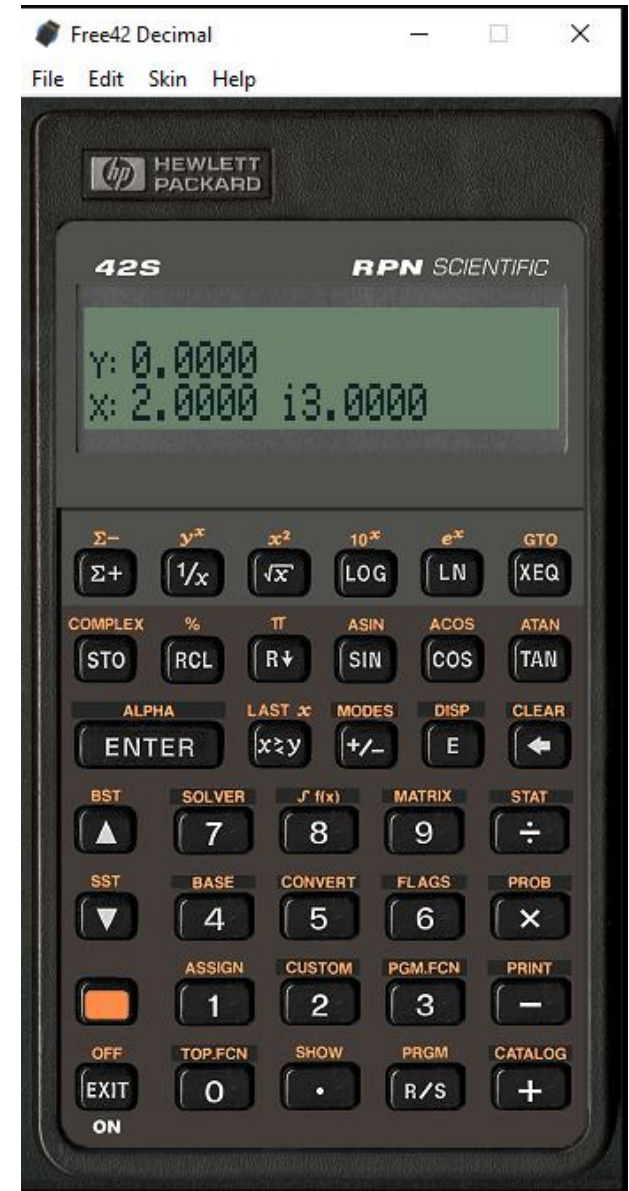
- the y register becomes the real part
- the x register becomes the complex part.

To input the number $2 + j3$, press

```
MODES  
RECT  
2  
enter  
3  
complex
```

In polar mode (yellow - MODES - POLAR)

- the y register becomes the magnitude
- the x register becomes the angle (in the current units degrees / rad / grad)



To input the number $5 \angle 27^\circ$

MODES

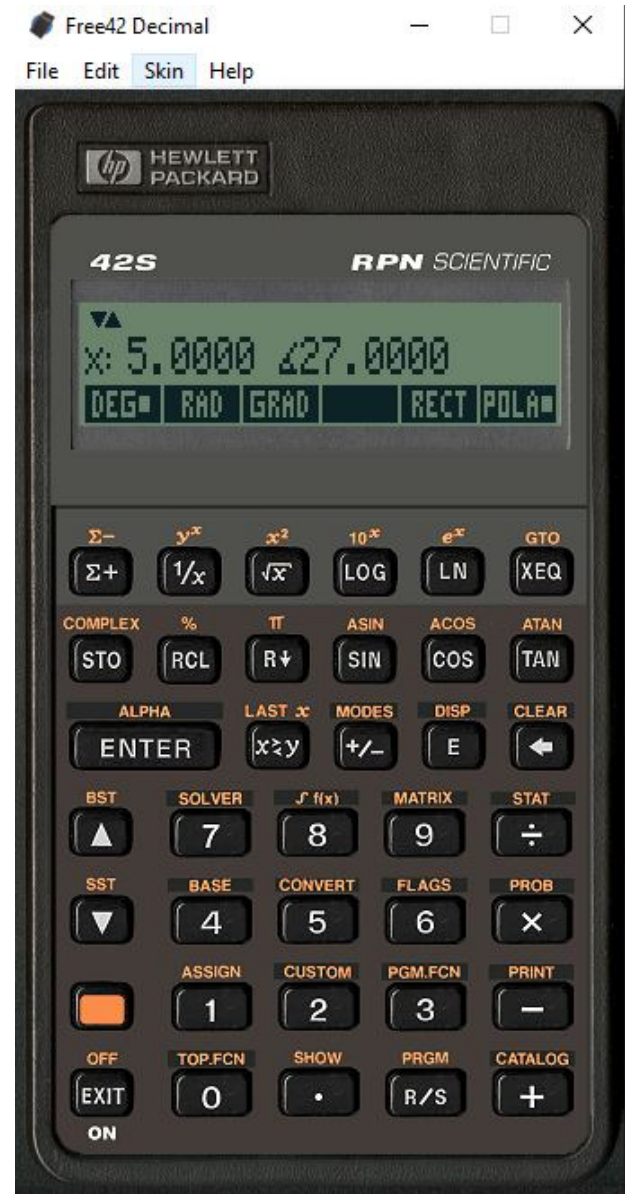
POLAR

5

enter

27

complex



Addition, Subtraction, Multiplication, and Division

With complex numbers, you can add, subtract, multiply, and divide just like real numbers:

Addition: For addition

- The real parts add, and
- The complex parts add.

Example:

$$(a_1 + jb_1) + (a_2 + jb_2) = (a_1 + a_2) + j(b_1 + b_2)$$

Subtraction: Again, the real parts subtract and the complex parts subtract

$$(a_1 + jb_1) - (a_2 + jb_2) = (a_1 - a_2) + j(b_1 - b_2)$$

Note: Addition and subtraction also work in polar form.

- This requires a polar to rectangular conversion:

$$\begin{aligned}r_1 \angle \theta_1 + r_2 \angle \theta_2 &= (r_1 \cos \theta_1) + j(r_1 \sin \theta_1) + (r_2 \cos \theta_2) + j(r_2 \sin \theta_2) \\ &= (r_1 \cos \theta_1 + r_2 \cos \theta_2) + j(r_1 \sin \theta_1 + r_2 \sin \theta_2)\end{aligned}$$

Moral: Addition and subtraction is easier in rectangular form. Or use a calculator that can add and subtract complex numbers.

Multiplication: Multiplication is a little trickier, but the result is a complex number.

$$(a_1 + jb_1)(a_2 + jb_2) = a_1a_2 + ja_1b_2 + ja_2b_1 + j^2b_1b_2$$

Note that $j^2 = -1$

$$(a_1 + jb_1)(a_2 + jb_2) = (a_1a_2 - b_1b_2) + j(a_1b_2 + a_2b_1)$$

Polar form actually works better for multiplication

$$\begin{aligned} r_1 \angle \theta_1 \cdot r_2 \angle \theta_2 &= r_1 e^{j\theta_1} \cdot r_2 e^{j\theta_2} \\ &= r_1 r_2 e^{j\theta_1} e^{j\theta_2} \end{aligned}$$

Using the property

$$e^a e^b = e^{a+b}$$

gives

$$r_1 \angle \theta_1 \cdot r_2 \angle \theta_2 = r_1 r_2 e^{j\theta_1 + j\theta_2}$$

$$r_1 \angle \theta_1 \cdot r_2 \angle \theta_2 = r_1 r_2 \angle (\theta_1 + \theta_2)$$

When you multiply complex numbers

- The magnitude multiplies and
- The angles add

Complex Conjugate: The complex conjugate of a complex number is

$$(a + jb)^* \equiv a - jb$$

The complex conjugate has the property that the product of a complex number with its complex conjugate is a real number, equal to the magnitude squared:

$$(a + jb) \cdot (a - jb) = a^2 + b^2$$

Division: Division also results in a complex number but takes even more computations. It uses the complex conjugate of the denominator:

$$\begin{aligned} \left(\frac{a_1 + jb_1}{a_2 + jb_2} \right) &= \left(\frac{a_1 + jb_1}{a_2 + jb_2} \right) \left(\frac{a_2 - jb_2}{a_2 - jb_2} \right) \\ &= \left(\frac{(a_1 a_2 + b_1 b_2) + j(-a_1 b_2 + a_2 b_1)}{a_2^2 + b_2^2} \right) \\ &= \left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} \right) + j \left(\frac{-a_1 b_2 + a_2 b_1}{a_2^2 + b_2^2} \right) \end{aligned}$$

Polar form is again simpler for division

$$\begin{aligned}\left(\frac{r_1 \angle \theta_1}{r_2 \angle \theta_2}\right) &= \left(\frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}}\right) \\ &= \left(\frac{r_1}{r_2}\right) e^{j\theta_1} e^{-j\theta_2} \\ &= \left(\frac{r_1}{r_2}\right) e^{j\theta_1 - j\theta_2} \\ &= \left(\frac{r_1}{r_2}\right) \angle (\theta_1 - \theta_2)\end{aligned}$$

The division of complex numbers is

- The ratio of the magnitudes and
- The difference in the angles.

Handout: Solve the following problems

Determine the result of the following operations with complex numbers

$$\begin{array}{r} 2 + j3 \\ + 9 + j4 \\ \hline \end{array} \quad \begin{array}{r} 2 + j3 \\ - 9 + j4 \\ \hline \end{array}$$

$$\begin{array}{r} 2 + j3 \\ * 9 + j4 \\ \hline \end{array}$$

$$\frac{2 + j3}{9 + j4}$$

Sample Problems

Problem 1: Find y

$$y = \left(\frac{(2+j3)(4+j5)+(6+j7)}{8+j9} \right)$$

By Hand: (pretty painful)

$$\begin{aligned}y &= \left(\frac{(2+j3)(4+j5)+(6+j7)}{8+j9} \right) \\&= \left(\frac{((8-15)+j(12+10))+6+j7}{(8+j9)} \right) \\&= \left(\frac{(-7+j22)+6+j7}{(8+j9)} \right) \\&= \left(\frac{-1+j29}{8+j9} \right) \left(\frac{8-j9}{8-j9} \right) \\&= \left(\frac{(-8+261)+j(232+9)}{64+81} \right) \\&= \left(\frac{253+j241}{145} \right) \\&= \left(\frac{253}{145} \right) + j \left(\frac{241}{145} \right) \\&= 1.7448 + j1.6621\end{aligned}$$

Using Matlab

$$y = \left(\frac{(2+j3)(4+j5)+(6+j7)}{8+j9} \right)$$

```
>> y = ( (2+j*3) * (4+j*5) + (6+j*7) ) / (8 + j*9)
```

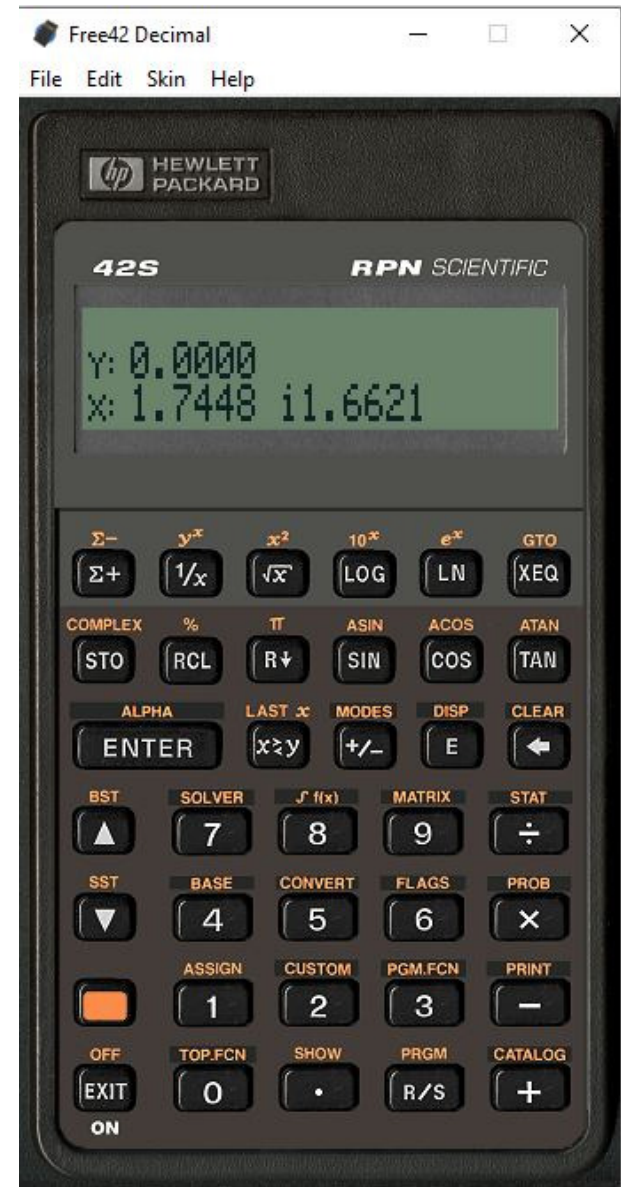
```
y = 1.7448 + 1.6621i
```



Using an HP42

$$y = \left(\frac{(2+j3)(4+j5)+(6+j7)}{8+j9} \right)$$

```
2
enter
3
complex
4
enter
6
complex
*
6
enter
7
complex
+
8
enter
9
complex
/
ans = 1.7448 + j1.6621
```



Partial Fraction Expansion with Real Poles

A common problem in ECE is to expand a function by its roots. For example, find

$$\left(\frac{2x+3}{(x+1)(x+2)(x+3)} \right) = \left(\frac{a}{x+1} \right) + \left(\frac{b}{x+2} \right) + \left(\frac{c}{x+3} \right)$$

Solution #1: (the hard way) Place the right side over a common denominator and match coefficients.

$$= \left(\frac{a}{x+1}\right)\left(\frac{(x+2)(x+3)}{(x+2)(x+3)}\right) + \left(\frac{b}{x+2}\right)\left(\frac{(x+1)(x+3)}{(x+1)(x+3)}\right) + \left(\frac{c}{x+3}\right)\left(\frac{(x+1)(x+2)}{(x+1)(x+2)}\right)$$

This places all terms over a common denominator. The numerator is then

$$2x + 3 = a(x + 2)(x + 3) + b(x + 1)(x + 3) + c(x + 1)(x + 2)$$

$$2x + 3 = a(x^2 + 5x + 6) + b(x^2 + 4x + 3) + c(x^2 + 3x + 2)$$

This gives three equations for three unknowns.

Matching the x^2 terms:

$$0 = a + b + c$$

x^1 terms:

$$2 = 5a + 4b + 3c$$

x^0 terms:

$$3 = 6a + 3b + 2c$$

Place in matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 5 & 4 & 3 \\ 6 & 3 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$$

Solve in Matlab

```
>> A = [1, 1, 1; 5, 4, 3; 6, 3, 2]
      1      1      1
      5      4      3
      6      3      2
```

```
>> B = [0; 2; 3]
      0
      2
      3
```



>> inv(A) * B

a 0.5000

b 1.0000

c -1.5000

SO

$$\left(\frac{2x+3}{(x+1)(x+2)(x+3)} \right) = \left(\frac{0.5}{x+1} \right) + \left(\frac{1}{x+2} \right) + \left(\frac{-1.5}{x+3} \right)$$



Solution #2: (cover-up method).

Equals is a powerful symbol: it means both sides are identical everywhere.

- The right side blows up (goes to infinity) near $x = \{-1, -2, -3\}$. The left side has to match.

Near $x = -1$, only the first term matters since it's going to infinity while the other terms are finite. So

$$\lim_{x \rightarrow -1} \left(\frac{2x+3}{(x+1)(x+2)(x+3)} \right) = \lim_{x \rightarrow -1} \left(\frac{a}{x+1} \right)$$

Cancel (cover up) the $(x+1)$ term and evaluate

$$a = \left(\frac{2x+3}{(x+2)(x+3)} \right)_{x=-1} = 0.5$$

Near $x = -2$, only the second term (b) matters on the right.

$$\lim_{x \rightarrow -2} \left(\frac{2x+3}{(x+1)(x+2)(x+3)} \right) = \lim_{x \rightarrow -2} \left(\frac{b}{x+2} \right)$$

Cancel (cover up) the $(x+1)$ term and evaluate

$$b = \left(\frac{2x+3}{(x+1)(x+3)} \right)_{x=-2} = 1$$

Near $x = -3$, only the third term (c) matters:

$$\lim_{x \rightarrow -3} \left(\frac{2x+3}{(x+1)(x+2)(x+3)} \right) = \lim_{x \rightarrow -3} \left(\frac{c}{x+3} \right)$$

$$c = \left(\frac{2x+3}{(x+1)(x+2)} \right)_{x=-3} = -1.5$$

Either method works - the cover-up method is a lot easier.

The cover-up method also works in Matlab.

- Take a number close to the point you're evaluating (perturb by $1e-9$)
- Solve for $\{a, b, c\}$

```
>> x = -1 + 1e-9;  
>> a = (2*x + 3) / ( (x+1) * (x+2) * (x+3) ) * (x+1)
```

```
a = 0.5000
```

```
>> x = -2 + 1e-9;  
>> b = (2*x + 3) / ( (x+1) * (x+2) * (x+3) ) * (x+2)
```

```
b = 1.0000
```

```
>> x = -3 + 1e-9;  
>> c = (2*x + 3) / ( (x+1) * (x+2) * (x+3) ) * (x+3)
```

```
c = -1.5000
```

Handout: Find the partial fraction expansion

$$\left(\frac{10(x+2)}{(x+5)x(x+10)} \right) = \left(\frac{a}{x+5} \right) + \left(\frac{b}{x} \right) + \left(\frac{c}{x+10} \right)$$



Partial Fraction with Complex Numbers

Placing all terms over a common denominator works, but is that much harder with complex numbers. The cover-up method is the same either way.

Example: Determine {a, b, c}

$$\left(\frac{5x+7}{(x+1+j3)(x+1-j3)(x+5)} \right) = \left(\frac{a}{x+1+j3} \right) + \left(\frac{b}{x+1-j3} \right) + \left(\frac{c}{x+5} \right)$$

Solving

$$a = \left(\frac{5x+7}{(x+1-j3)(x+5)} \right)_{x=-1-j3} = 0.3600 + j0.3533$$

$$b = \left(\frac{5x+7}{(x+1+j3)(x+5)} \right)_{x=-1+j3} = 0.3600 - j0.3533$$

$$c = \left(\frac{5x+7}{(x+1+j3)(x+1-j3)} \right)_{x=-5} = -0.7200$$

Solving using Matlab

```
>> x = -1 - j*3 + 1e-9;  
>> a = (5*x+7) / ( (x+1+j*3)*(x+1-j*3)*(x+5) ) * (x+1+j*3)  
  
a = 0.3600 + 0.3533i
```

```
>> x = -1 + j*3 + 1e-9;  
>> a = (5*x+7) / ( (x+1+j*3)*(x+1-j*3)*(x+5) ) * (x+1-j*3)  
  
a = 0.3600 - 0.3533i
```

```
>> x = -5 + 1e-9;  
>> a = (5*x+7) / ( (x+1+j*3)*(x+1-j*3)*(x+5) ) * (x+5)  
  
a = -0.7200
```

More Fun with Complex Numbers

Note: These use the properties

$$\ln(e^x) = x$$

$$e^{\ln(x)} = x$$

$$(e^a)^b = e^{ab}$$

Example 1: Find y :

$$y = (2 + j3)^{(4+j5)}$$



Solution: Convert to polar form (using radians)

$$\begin{aligned}2 + j3 &= 3.6056 \angle 0.9828 \\ &= e^{\ln(3.6056)} \cdot e^{j0.9828} \\ &= e^{1.2825 + j0.9828}\end{aligned}$$

Raise to the power

$$\begin{aligned}&= (e^{1.2825 + j0.9828})^{(4 + j5)} \\ &= e^{(0.2159 + j10.3453)}\end{aligned}$$

Separate

$$\begin{aligned}&= e^{0.2159} e^{j10.3435} \\ &= 1.2410 \cdot (\cos(10.3425) + j \sin(10.3435)) \\ &= 1.2410 \cdot (-0.6068 - j0.7942) \\ &= -0.7530 - j0.9864\end{aligned}$$

Check in Matlab

```
>> y = (2 + j*3) ^ (4 + j*5)
```

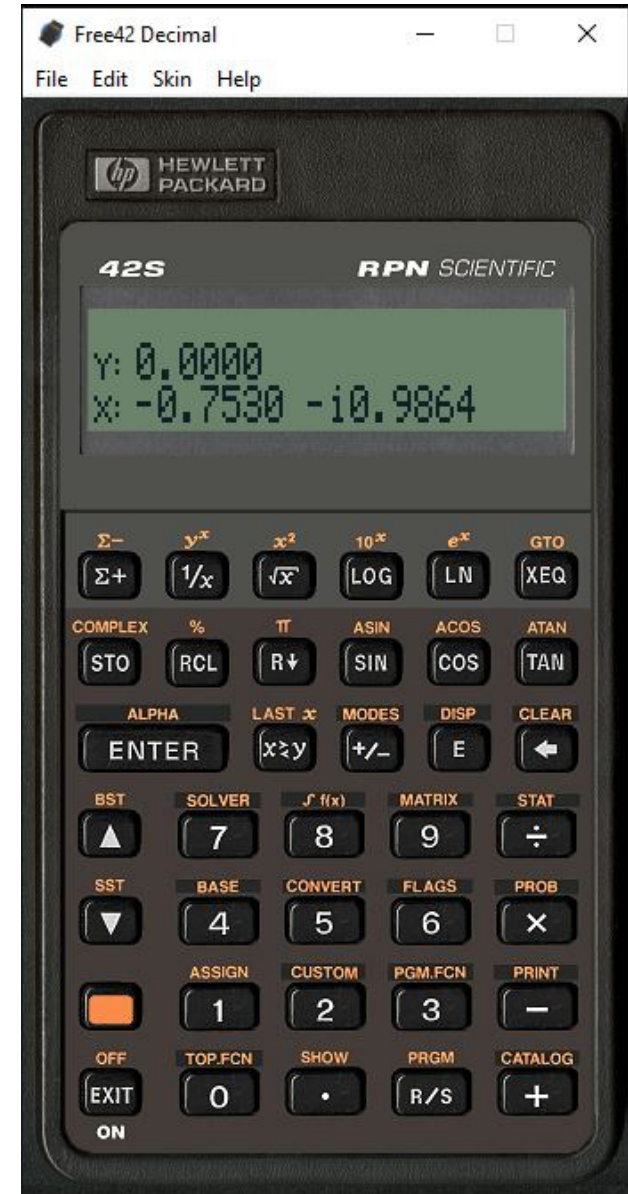
```
y = -0.7530 - 0.9864i
```

Note: Matlab was *way* easier.

Check on an HP42:

```
2  
enter  
3  
complex  
4  
enter  
5  
complex  
y^x
```

ans = $-0.7530 - j0.9864$



Example 2: Find y:

$$y = \cos(2 + j3)$$

Use Euler's identity

$$\begin{aligned}\cos(2 + j3) &= \left(\frac{1}{2}\right) \left(e^{(2+j3)} + e^{j(2+j3)}\right) \\ &= \left(\frac{1}{2}\right) \left((e^2 e^{j3}) + (e^{j2} e^{-3})\right) \\ &= \left(\frac{1}{2}\right) \left(e^2 \cdot (\cos(3) + j \sin(3)) + e^{-3} \cdot (\cos(2) + j \sin(2))\right) \\ &= \left(\frac{e^2 \cos(3) + e^{-3} \cos(2)}{2}\right) + j \left(\frac{e^2 \sin(3) + e^{-3} \sin(2)}{2}\right)\end{aligned}$$

Check in Matlab

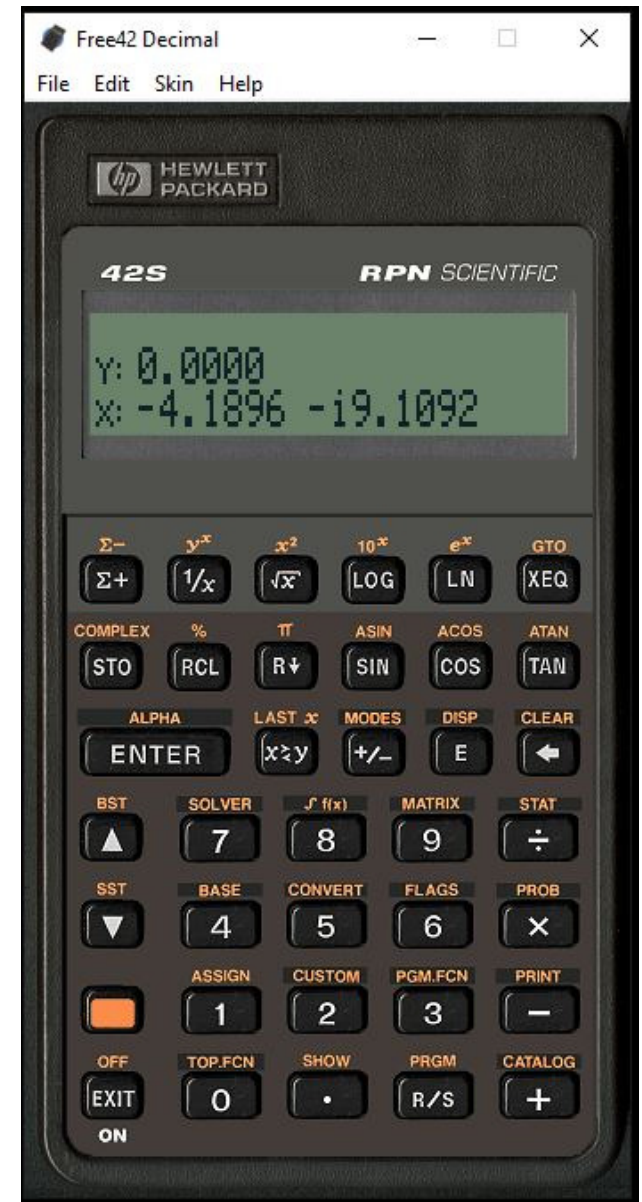
```
>> cos(2 + j*3)
```

```
ans = -4.1896 - 9.1092i
```

Check on an HP42

```
2  
enter  
3  
complex  
COS
```

```
ans = -4.1896 - j9.1092
```



Find y :

$$y = \ln(2 + j3)$$

Express as an exponential

$$\begin{aligned} 2 + j3 &= 3.6056 \angle 0.9828 \\ &= e^{\ln(3.6056) + j0.9828} \end{aligned}$$

$$\ln(2 + j3) = \ln(3.6056) + j0.9828$$

Check in Matlab

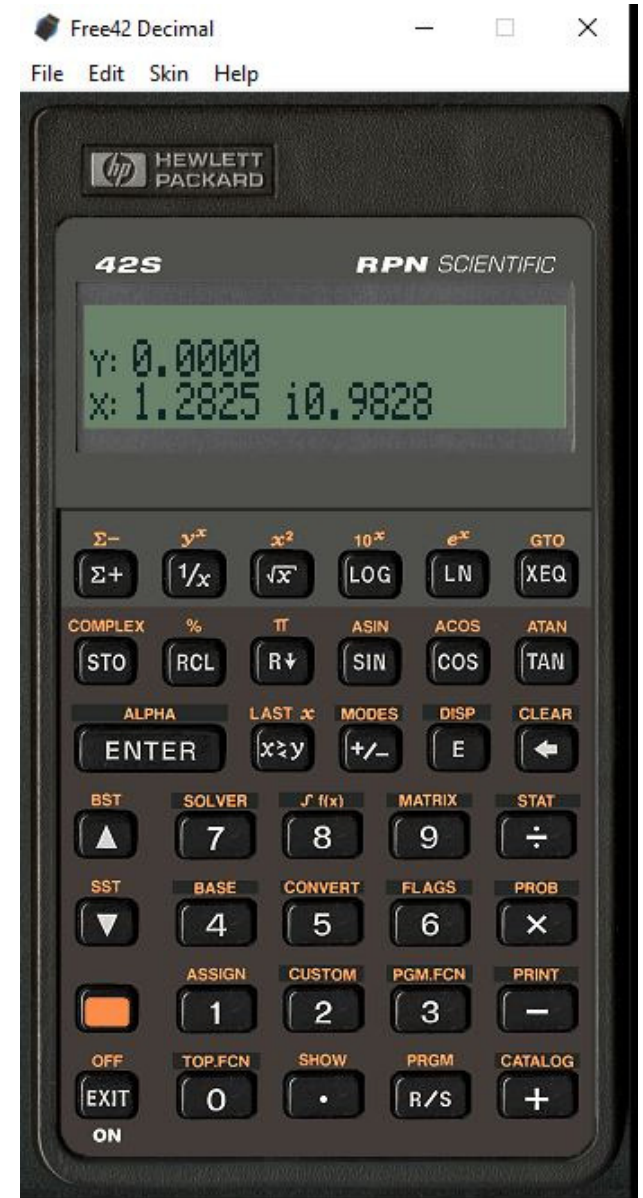
```
>> log(2 + j*3)
```

```
ans = 1.2825 + 0.9828i
```

Check with an HP42

```
2  
enter  
3  
ln
```

```
ans = 1.2825 + 0.9828i
```



Handout:

- Find y by hand
- Check your answer with Matlab or a calculator

$$y = \sqrt{i}$$

Moral #1: Pretty much anything you do with real numbers you can do with complex numbers. The answer will be complex though.

Moral #2: When dealing with complex numbers, it is a *lot* easier to use Matlab or an HP calculator than doing it by hand...

Summary:

- Complex numbers allow you to represent something with two degrees of freedom
 - Addition, subtraction, multiplication, and division all work with complex numbers
 - Calculations using complex numbers by hand is really painful.
 - Calculations using complex numbers aren't too bad with Matlab or an HP calculator
 - Get familiar with a calculator which does complex numbers before you get to Circuits I
 - You're going to need it for midterms for pretty much all courses in ECE
-