EE 206: Circuits I

Topics

- Kirchoff's Current Loops (KCL)
- Kirchoff's Voltage Nodes (KVN)
- Writing N equations for N unknowns
- Solving N equations for N unknowns
- Solving circuits using CircuitLab (www.CircuitLab.com)

Matlab Functions

- Input a matrix
- Solving N equations for N unknowns

Resistor Networks & Steady-State Heat Flow

A common theme in solving problems in Electrical and Computer Engineering is to first write N equations to solve for N unknowns. Once done, you can usually solve.

One Dimensional Heat Flow:

Suppose you want to know the temperature along a long rod with a fixed temperature at one end
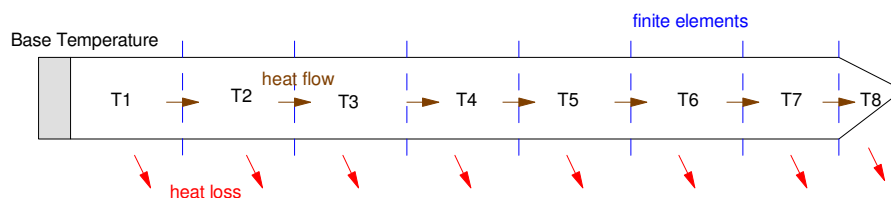
One-dimensional heat flow. Heat flows left to right - with heat loss at each element

One way to solve this problem is to split the rod into a large number of finite element.

- Each element has a temperature
- Between elements is thermal conduction.

The steady-state solution is when the heat flowing into a node equals the heat flow out of a node.

The circuit equivalent for this heat-flow problem is a resistor circuit:

- Voltages represent the temperature at each point along the rod and current represents heat flow
- The resistors on the top model the thermal conductivity between nodes
- The resistors to ground model the heat loss at each node.
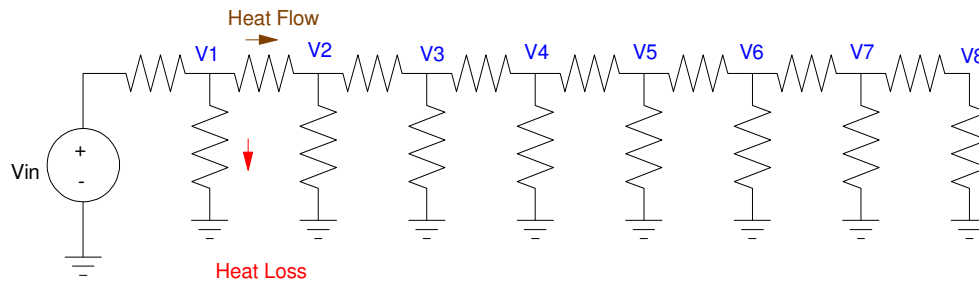
Circuit model for 1-dimensional heat flow: voltages represent temperature, current represents heat flow

If you can solve for the voltages for the resistor network, you know the temperatures along the rod.

So, this leads the following problems:

- How do you solve for the voltages in the above circuit?
  How do you solve for the temperatures at each point along a 1-dimensional metal rod?
- How do you solve for the currents in the above circuit
 *How do you solve for the heat flow along the length of a metal rod?*

Three methods are covered in Circuits I:

- Analysis: You can solve N equations for N unknowns where N is the number of voltages or currents you're trying to find,
- Simulation: You can input this circuit into program like CircuitLab and have it compute the voltages and currents
- Experiment: You can go into the lab, build the circuit, and take measurements.

In this class, we'll just cover the first two methods.

To illustrate these methods, consider the following resistor circuit and the problem of:

- Finding the voltages, and
- Finding the currents
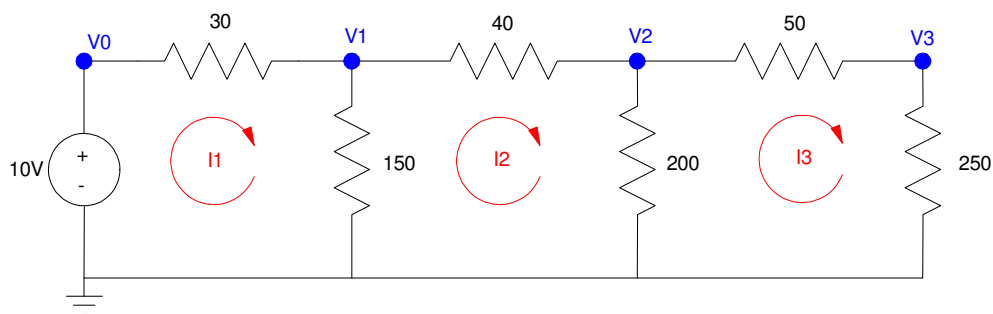
Circuit Problem: Determine the voltages and/or currents

Method #1: Voltage Nodes (KVN)

The idea behind this method is this:

- At equilibrium, the current going into each node must match the current leaving the node.
- The voltages at each node are whatever it takes to produce equilibrium (current in = current out)

The procedure for KVN is as follows:

Step 1: Define one node to be the ground reference. Then, define and label the nodes of the circuit.
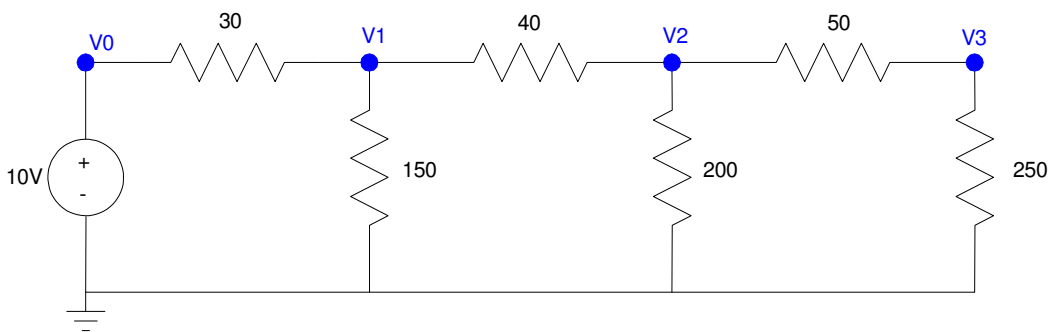
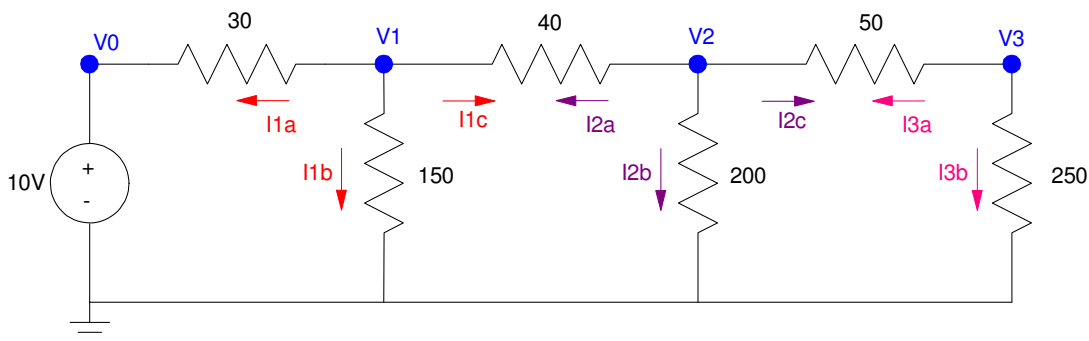Step #1 for KVN: Define circuit ground and label each voltage node.

Step 2: Write N equations to solve for N unknown voltages.

Since there are four voltage nodes, we need 4 equations to solve for 4 unknown voltages.

Node V0 is easy: it's voltage is defined by the 10V source

$$V_0 = 10$$

For the other three equations, sum the currents from the node to zero (conservation of current).

Step 2: Write 3 more equations by summing the current from the node to zero

At node V1:

$$I_{a1} + I_{b1} + I_{c1} = 0$$

$$\left(\frac{V_1 - V_0}{30}\right) + \left(\frac{V_1}{150}\right) + \left(\frac{V_1 - V_2}{40}\right) = 0$$

At node V2:

$$I_{a2} + I_{b2} + I_{c2} = 0$$

$$\left(\frac{V_2 - V_1}{40}\right) + \left(\frac{V_2}{200}\right) + \left(\frac{V_2 - V_3}{50}\right) = 0$$

At node V3:

$$I_{a3} + I_{b3} = 0$$

$$\left(\frac{V_3 - V_2}{50}\right) + \left(\frac{V_3}{250}\right) = 0$$

So, the 4 equations for 4 unknowns are:

$$V_0 = 10$$

$$\left(\frac{V_1 - V_0}{30}\right) + \left(\frac{V_1}{150}\right) + \left(\frac{V_1 - V_2}{40}\right) = 0$$

$$\left(\frac{V_2 - V_1}{40}\right) + \left(\frac{V_2}{200}\right) + \left(\frac{V_2 - V_3}{50}\right) = 0$$

$$\left(\frac{V_3 - V_2}{50}\right) + \left(\frac{V_3}{250}\right) = 0$$

Sidelight: One way to check for sign errors is as follows:

- At node V1, all the coefficients of V1 are positive whereas all other terms are negative
- Likewise at node V2 and V3

This pattern (always?) holds with voltage nodes.

Step 3: Solve N equations for N unknowns.

First, group terms

$$V_0 = 10$$

$$\left(\frac{-1}{30}\right) V_0 + \left(\frac{1}{30} + \frac{1}{150} + \frac{1}{40}\right) V_1 + \left(\frac{-1}{40}\right) V_2 = 0$$

$$\left(\frac{-1}{40}\right) V_1 + \left(\frac{1}{40} + \frac{1}{200} + \frac{1}{50}\right) V_2 + \left(\frac{-1}{50}\right) V_3 = 0$$

$$\left(\frac{-1}{50}\right) V_2 + \left(\frac{1}{50} + \frac{1}{250}\right) V_3 = 0$$

Place in matrix form

$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\
\left(\frac{-1}{30}\right) & \left(\frac{1}{30} + \frac{1}{150} + \frac{1}{40}\right) & \left(\frac{-1}{40}\right) & 0 \\
0 & \left(\frac{-1}{40}\right) & \left(\frac{1}{40} + \frac{1}{200} + \frac{1}{50}\right) & \left(\frac{-1}{50}\right) \\
0 & 0 & \left(\frac{-1}{50}\right) & \left(\frac{1}{50} + \frac{1}{250}\right)
\end{bmatrix}
\begin{bmatrix}
V_0 \\ V_1 \\ V_2 \\ V_3
\end{bmatrix}
=
\begin{bmatrix}
10 \\ 0 \\ 0 \\ 0
\end{bmatrix}
$$

Solve (Matlab helps here)

```
A = [1,0,0,0 ; -1/30,1/30+1/150+1/40,-1/40,0];
A = [A ; 0,-1/40,1/40+1/200+1/50,-1/50 ; 0,0,-1/50,1/50+1/250]

    1.0000         0         0         0
   -0.0333    0.0650   -0.0250         0
         0   -0.0250    0.0500   -0.0200
         0         0   -0.0200    0.0240

B = [10;0;0;0]

    10
     0
     0
     0

V = inv(A)*B

V0    10.0000
V1     7.2072
V2     5.4054
V3     4.5045
```

Using CircuitLab, you can check your answers:

Validation in CircuitLab (www.CircuitLab.com):

There are many circuit simulation packages. We are currently using Circuitlab in our department since

- It works
- It's fairly easy to use,
- The answers are correct, and
- It's free to use for NDSU students (use your NDSU email when registering)
- It's $24/year otherwise

The method to determine the voltages in CircuitLab is as follows:

First, go the menu items on the left and add the voltage source and resistors.
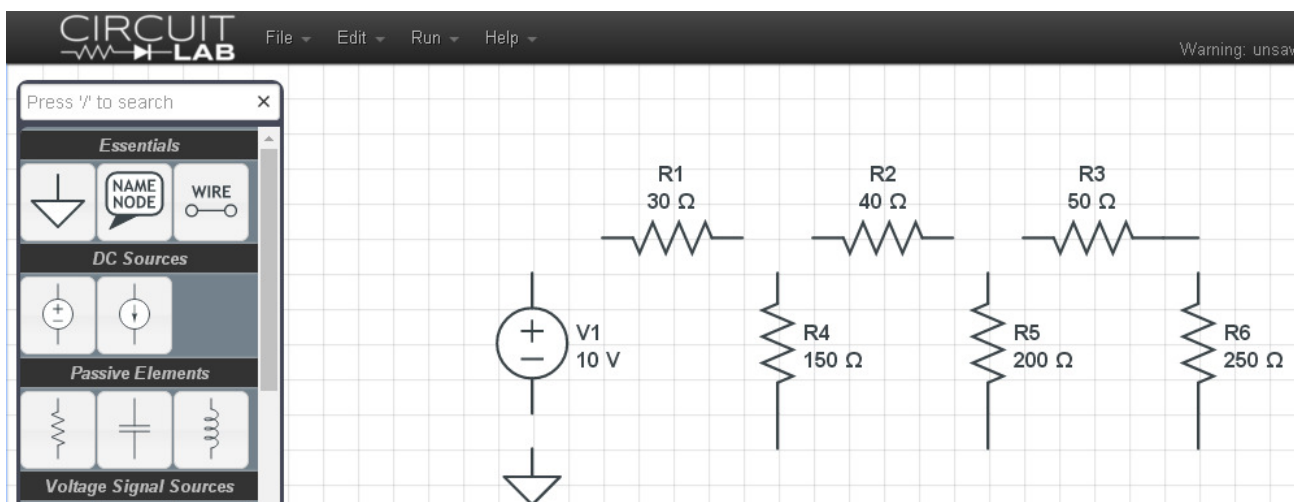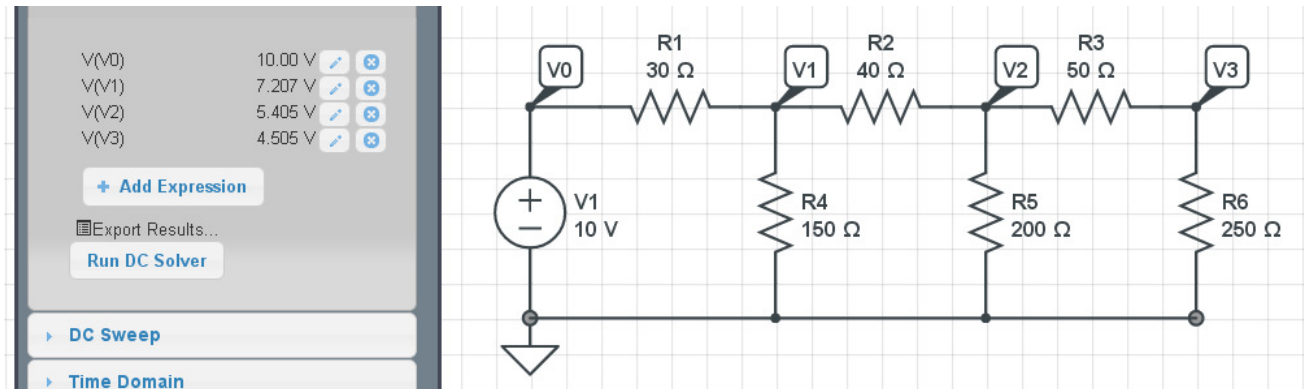

CircuitLab: Drag and drop components. Double click to change values

Connect the components by clicking on one end of an element and dragging the wire to another element. Once done, change the values to match the previous circuit

Once connected, you can solve for the voltages by clicking

- Add Expression: click on V0, V1, V2, V3
- Simulate
- DC Solver

CircuitLab: DC Solver for node voltages

Note that the solution matches our calculations

```
V = inv(A)*B

V0    10.0000
V1     7.2072
V2     5.4054
V3     4.5045
```

Method #2: Current Loops

Current Loops is based upon conservation of volage: the sum of the voltages around any closed loop must be zero. To use this method

Step 1) Draw the circuit so that there are N distinct "windows". Define the current in each window.
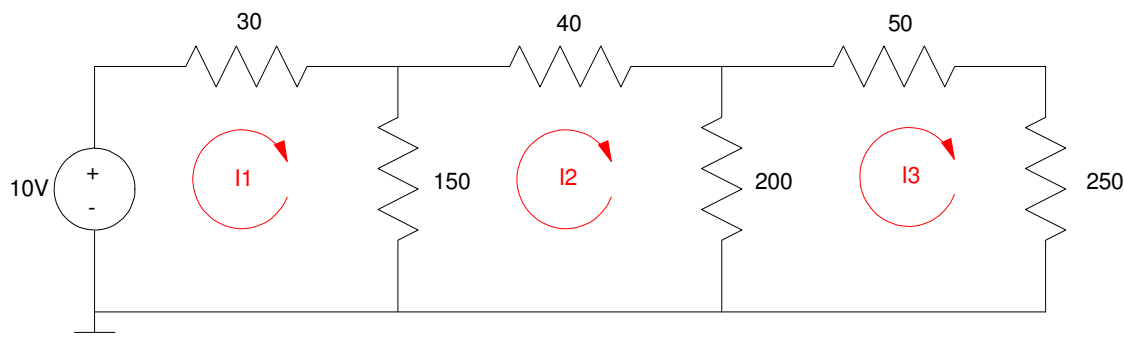
Step 2) Sum the voltages around each loop to zero to create N equations for the N unknown currents.

- If you pass through a voltage source, add if you hit the + terminal first, subtract if you hit the - terminal first.

Step 3) Solve N equations for N unknows.

For example, determine the currents for the previous circuit:

Step 1: This circuit has three distinct windows. Define the current around each loop to be { I1, I2, I3 }.

Current Loops: Define the current (name and direction) in each window

Step 2: Write N equations for N unkowns.

There are three unknown currents, so you need three equations. Summing the voltages around each loop:

Loop #1:

$$-10 + 30I_1 + 150(I_1 - I_2) = 0$$

Loop #2:

$$150(I_2 - I_1) + 40I_2 + 200(I_2 - I_3) = 0$$

Loop #3

$$200(I_3 - I_2) + 50I_3 + 250I_3 = 0$$

Sidelight:

- Note that for loop #1, all the I1 terms are positive and all other terms are negative.
- The same holds for the other loop equations.

Step 3: Solve.

Group terms

$$180I_1 - 150I_2 = 10$$

$$-150I_1 + 390I_2 - 200I_3 = 0$$

$$-200I_2 + 500I_3 = 0$$

Place in matrix form

$$\begin{bmatrix} 180 & -150 & 0 \\ -150 & 390 & -200 \\ 0 & -200 & 500 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

Solve using Matlab

```
>> A = [180,-150,0 ; -150,390,-200 ; 0,-200,500]

   180  -150     0
  -150   390  -200
     0  -200   500

>> B = [10;0;0]

    10
     0
     0

>> I = inv(A)*B

I1    0.0931
I2    0.0450
I3    0.0180
```
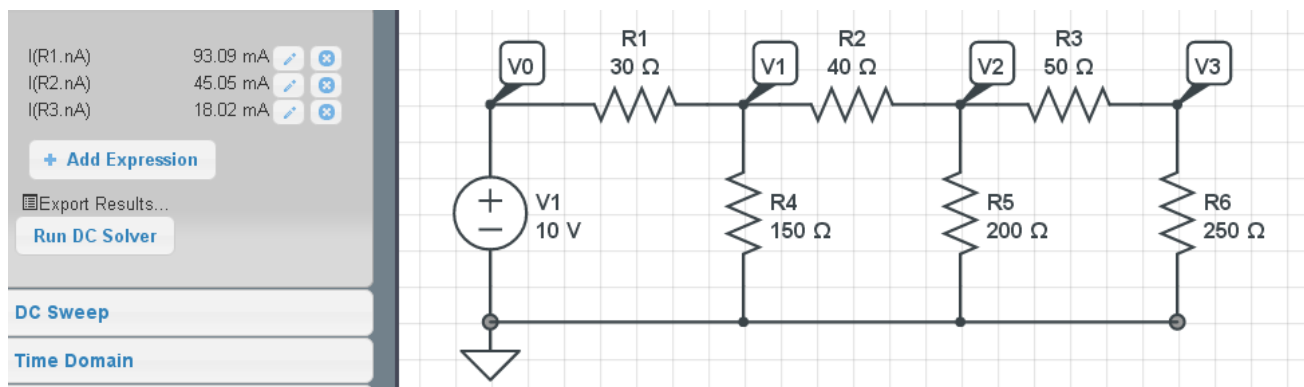
Validation: Using CircuitLab, you can check your answers. To select a current, click where the current goes into the resistor

Currents computed from CircuitLab match our calculations.

Calculations:

```
I1    0.0931
I2    0.0450
I3    0.0180
```

Sidelight

Back in the 1970's, solving 4 equations for 4 unknowns was a pain. 5 equations for 5 unknowns was *really* painful. We likewise would do all sorts of tricks to reduce the number of equations we would try to solve.

Today with Matlab, you don't really care how many equations you have: Matlab can solve 20 equations for 20 unknowns just as easily as 2 equations for 2 unknowns. Likewise, what's most important today is to get the equations correct.

For example, solve for the voltages for the following 10-stage resistor circuit:
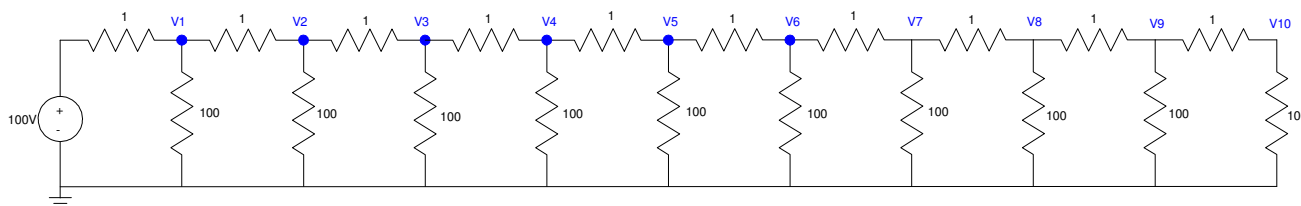
This looks painful but it's really not that bad. Since the resistors attached to each node are all the same, the equation at one node will be the same as any other node (except the last node). For example, the node equation at V5 would be

$$\left(\frac{V_5 - V_4}{1}\right) + \left(\frac{V_5 - V_6}{1}\right) + \left(\frac{V_5}{100}\right) = 0$$

or

$$-V_4 + 2.01 V_5 - V_6 = 0$$

The same pattern will hold for all 10 elements, except for the last one (where there is only one 1 Ohm resistor attached)

$$-V_9 + 1.01 V_{10} = 0$$

In matrix form, the equations will be

$$A \begin{bmatrix} V_1 \\ \vdots \\ V_{10} \end{bmatrix} = \begin{bmatrix} B_1 \\ \vdots \\ B_{10} \end{bmatrix}$$

In MATLAB, you can input this using a for statement

Start with a zero matrix with dimension 10x10

```
-->A = zeros(10,10);
```

The diagonal is -2.01. The off diagonals are +1

```
-->for i=1:9
-->   A(i,i)   = 2.01;
-->   A(i+1,i) = -1;
-->   A(i,i+1) = -1;
-->   end
```

The last element is 1.01

```
-->A(10,10) = 1.01;
```

Checking it it's correct:

```
--A

    2.01  - 1.     0.     0.     0.     0.     0.     0.     0.     0.
  - 1.     2.01  - 1.     0.     0.     0.     0.     0.     0.     0.
    0.    - 1.     2.01  - 1.     0.     0.     0.     0.     0.     0.
    0.     0.    - 1.     2.01  - 1.     0.     0.     0.     0.     0.
    0.     0.     0.    - 1.     2.01  - 1.     0.     0.     0.     0.
    0.     0.     0.     0.    - 1.     2.01  - 1.     0.     0.     0.
    0.     0.     0.     0.     0.    - 1.     2.01  - 1.     0.     0.
    0.     0.     0.     0.     0.     0.    - 1.     2.01  - 1.     0.
    0.     0.     0.     0.     0.     0.     0.    - 1.     2.01  - 1.
    0.     0.     0.     0.     0.     0.     0.     0.    - 1.     1.01
```
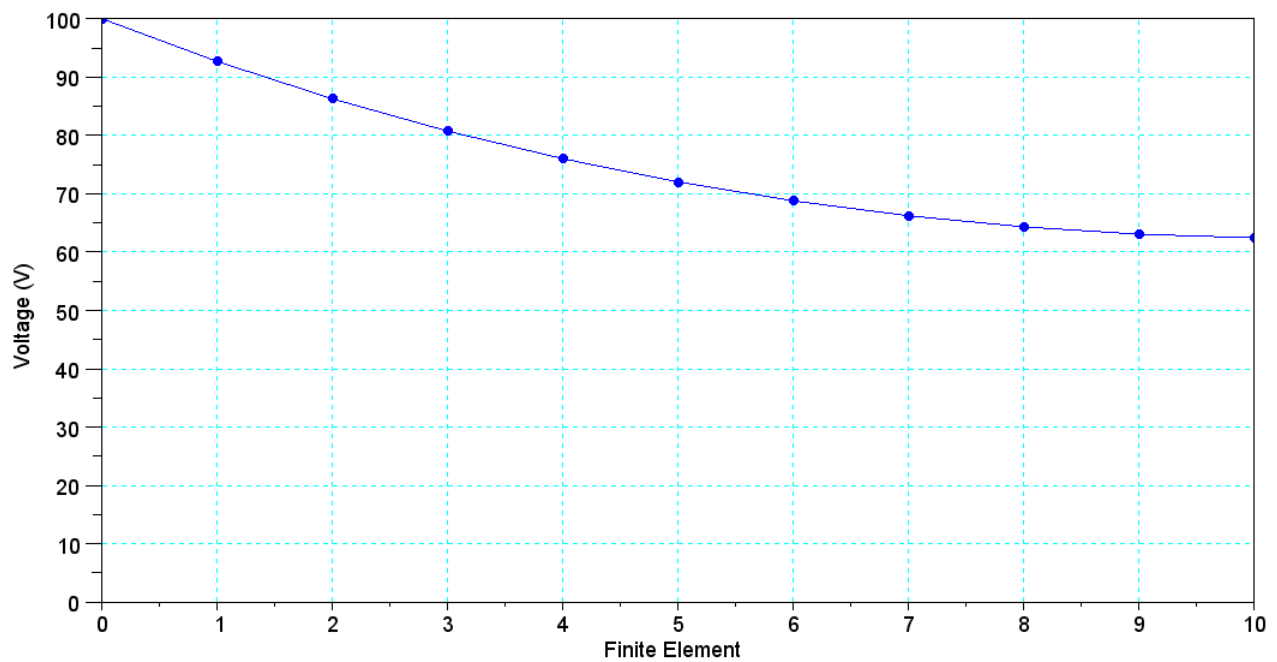
```
-->V = inv(A)*B

    92.673872
    86.274482
    80.737837
    76.008571
    72.03939
    68.790603
    66.229722
    64.331139
    63.075867
    62.451353
```

If you build this circuit in CircuitLab, you'll get the same result.

A graph shows the cooling of the bar as you go along a little better:

```
-->plot([0:10],[100;V],'.-')
-->xlabel('Finite Element');
-->ylabel('Voltage (V)');
```


Voltage at each node for a 10-stage resistor circuit.