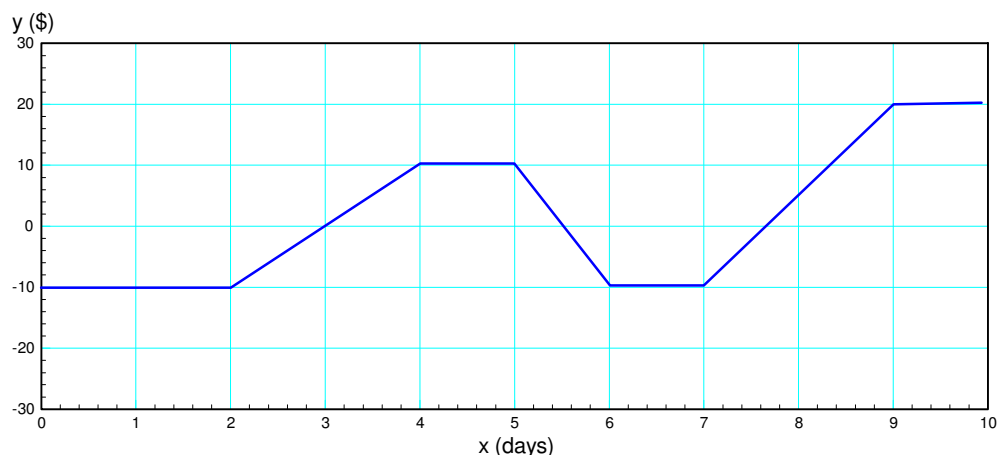# Math 165: Calculus I

Differentiation

Topics

- Differentiation
- Differentiation using Graphical Methods
- Numerical Differentiation
- Path Planning

Differentiation

The derivative of a function gives you the slope. This has several uses. For example, assume the following function is the balance of your bank account at any given time.

The derivative is the slope: this tells you what is happening with your bank account:

- When the balance remains constant (0<x<2), nothing is being deposited.
- When the balance increases (2<x<4), you are adding money (depositing money),
- When the balance decreases (5<x<6), you are withdrawing money.

Furthermore, the derivative also tells you how much money you're depositing or withdrawing.

To calculate the derivative at point, determine the slope:

$$slope = y'(x) = \frac{dy}{dx} = \frac{\text{change in y}}{\text{change in x}}$$

For example
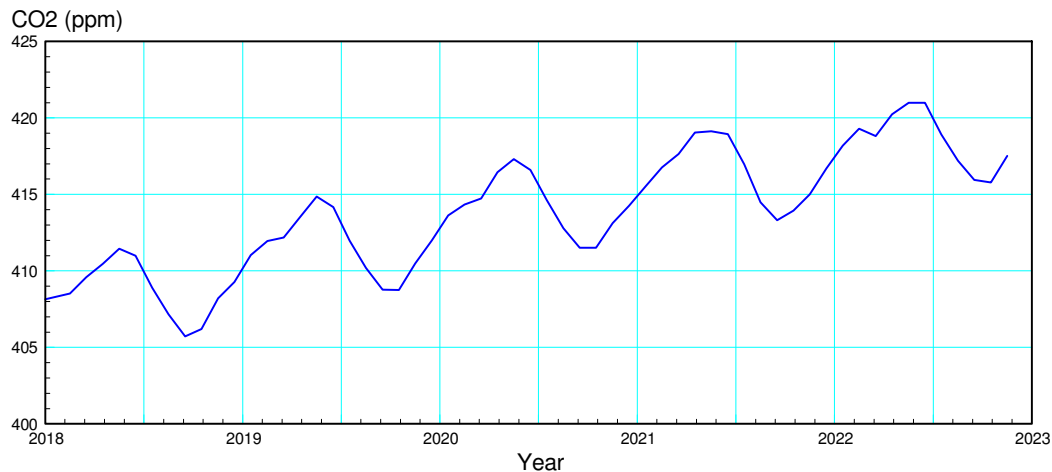
- For 0<x<2, y chances by zero. The slope is then 0/2 = 0
- For 2<x<4, y changes by +20 when x changes by +2. The slops is 20/2 = +10
- For 5<x<6, y changes by -20 as x changes by 1. The slops is -20/1 = -20
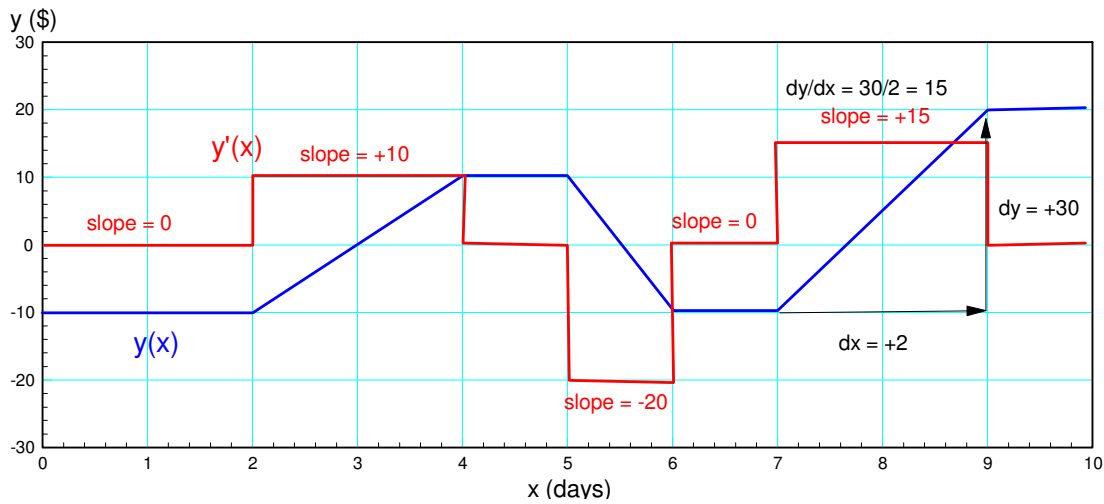
y(t) (blue) and its derivative y'(t) (read)

As another example, consider the global CO2 levels from 2015 - 2023 (shown below).

This graph shows the global CO2 levels. The slope (the derivative) tells you how much CO2 is being added to the atmosphere at any given time.:

- When the slope is positive, CO2 is being added to the atmosphere
- When the slope is negative, CO2 is being removed from the atmosphere

The derivative of y is shown below. What this graph tells you is

- During summer in the northern hemisphere, global CO2 levels drop. This is due to there being more land in the northern hemisphere than in southern. As trees and plants green up and grow during the northern summers, CO2 is absorbed.
- Starting in October, CO2 levels start to increase (y' is positive) as northern plants drop their leaves and get ready for the northern winter.
- CO2 levels keep increasing (y' remains positive) until May.

- I'm not exactly sure why, but many years see a drop in CO2 production in February and March, followed by an up-tick in April.

Derivatives are useful: they provide information about what's happening on a month-by-month basis.
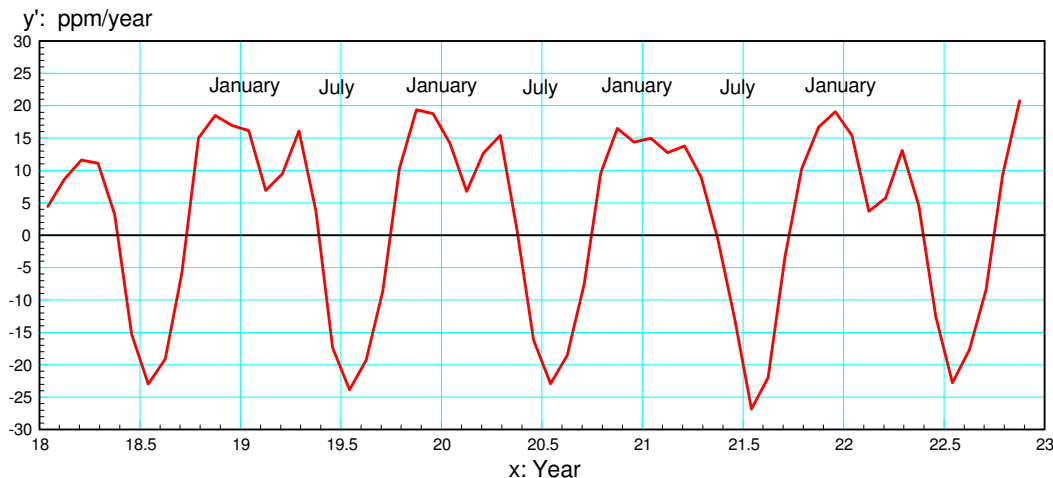
Derivative of Global CO2 Levels

In this lecture we'll look at several ways to compute the derivative of a function:
- Using lookup tables similar to what you'll do in Math 165,
- Using graphical methods, and
- Using numerical methods (i.e. Matlab).

We'll also look at using derivatives to evaluate different ways for a robotic arm to go from point A to point B.
- The derivative of the motor angle gives you the motor velocity - which is essentially the voltage applied to a DC motor.
- The second derivative of motor angle gives the motor acceleration - which is essentially the current required for this motor.

By looking at the derivative and second derivative of a proposed path from point A to B, the inputs to a DC motor can likewise be computed ahead of time.

Differentiation: Method #1

One way to take the derivative of function is to memorize and apply a set of rules. (This is what you do in Calculus I). Some common rules are

Power Functions:

$$\frac{d}{dx}(x^n) = n \cdot x^{n-1}$$

Exponential

$$\frac{d}{dx}(e^{ax}) = a \cdot e^{ax}$$

Sine Function

$$\frac{d}{dx}(\sin(ax)) = a \cdot \cos(ax)$$

Cosine Function

$$\frac{d}{dx}(\cos(ax)) = -a \cdot \sin(ax)$$

Chain Rule

$$\frac{d}{dx}(f(x) \cdot g(x)) = \frac{df(x)}{dx} \cdot g(x) + f(x) \cdot \frac{dg(x)}{dx}$$

Division

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) \cdot \frac{df}{dx} - f(x) \cdot \frac{dg}{dx}}{(g(x))^2}$$

With these rules, you can find the derivative of functions such as

$$y = \frac{d}{dx}\left(\frac{\cos(x)}{x^2+1}\right)$$

Stay tuned until you take Calculus I to solve using this method.

Differentiation Method #2: Graphical

A second way to find a derivative is to determine the slope at each point, or

- Determine the slope at several points, and then
- Draw a curve connecting these points.

Example 1: Using graphical methods, find the derivative of $y = x^2$

- First, draw the function
- Compute the slope (the derivative) at several points
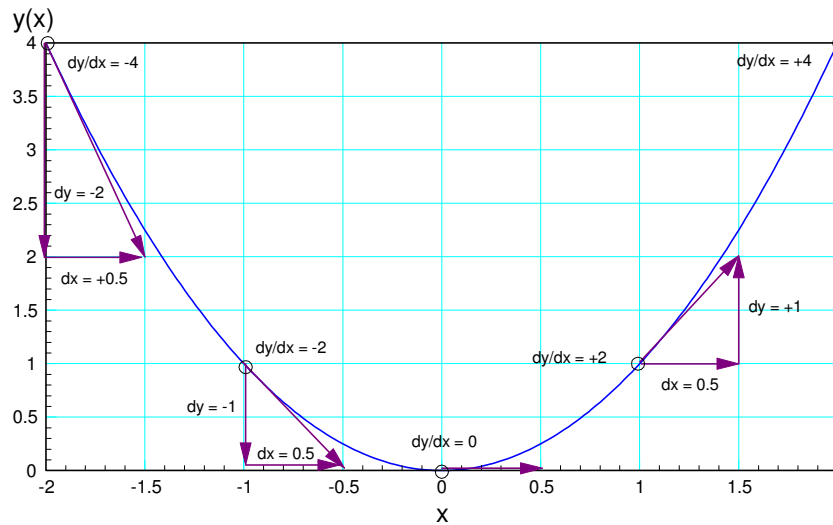
Plot the derivatives and connect with a smooth curve (shown in purple)
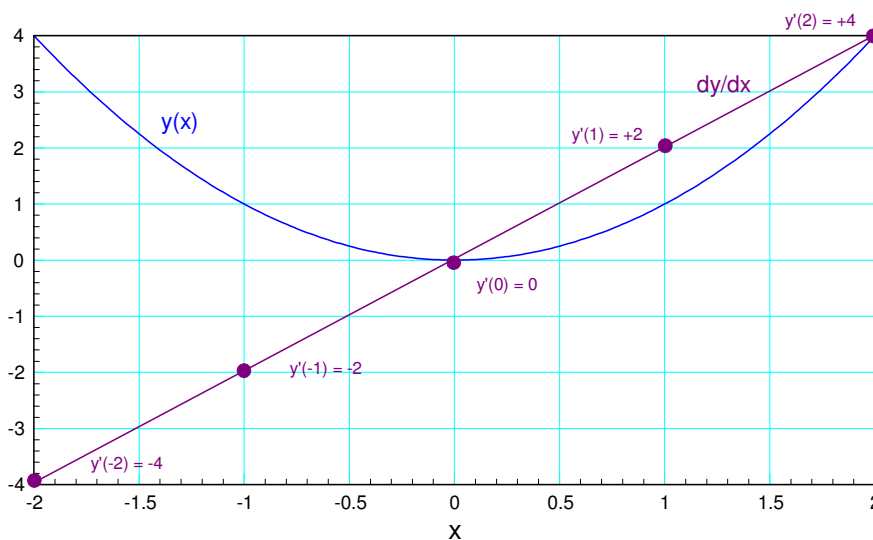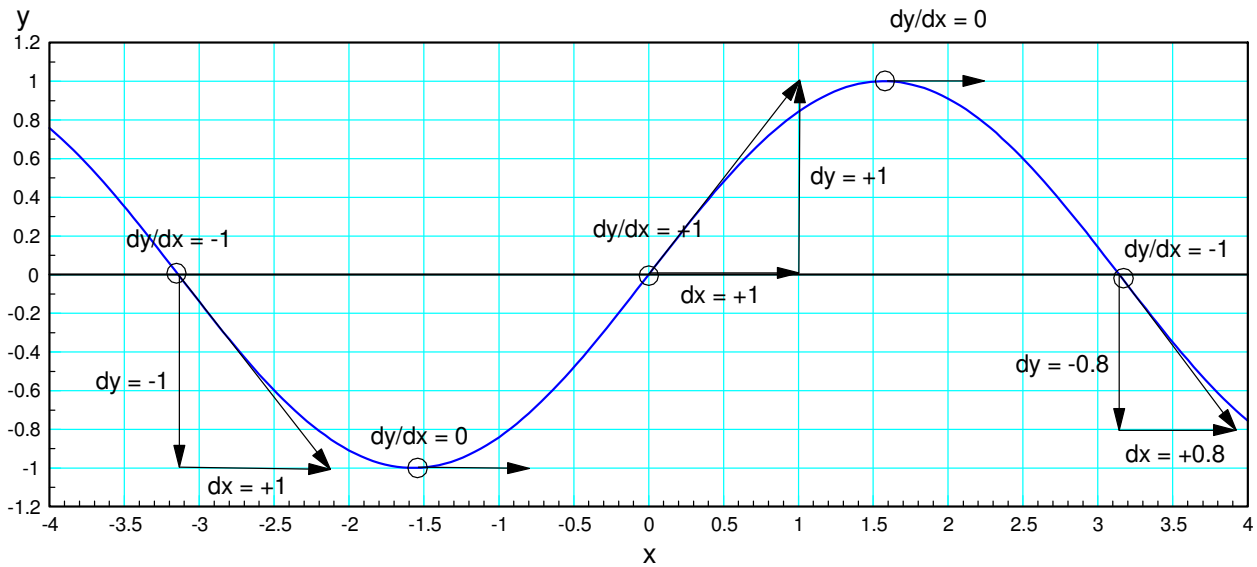
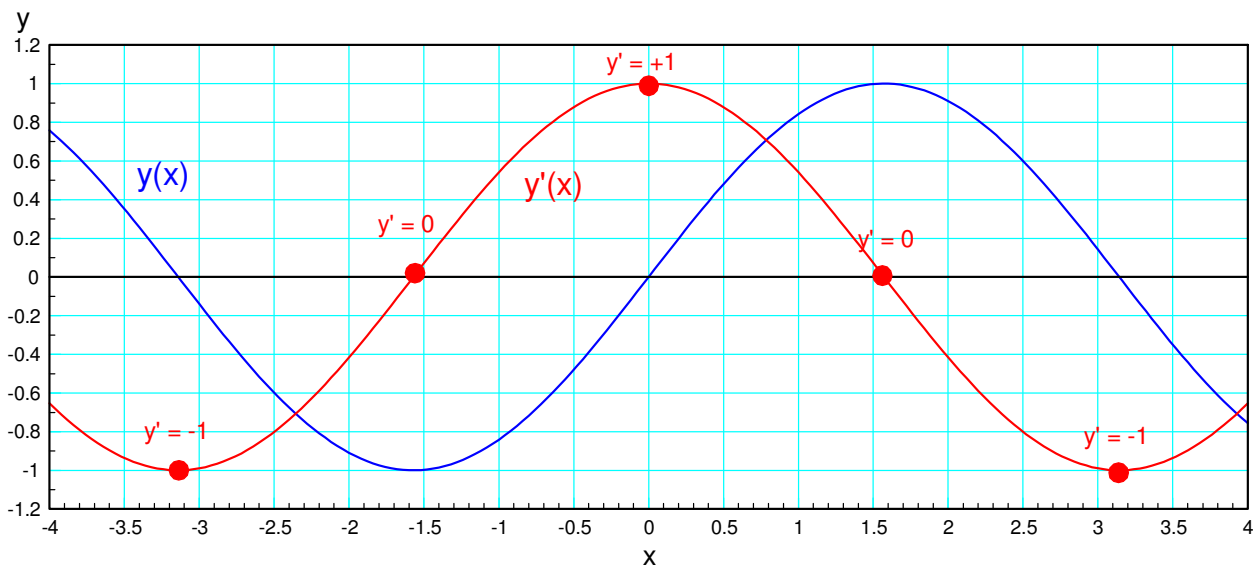Example #2: Determine the derivative of a sine wave using graphical methods:

$$y = \sin(x)$$

Step 1: Plot y vs. x (shown below)

Step 2: Compute the slope at several points

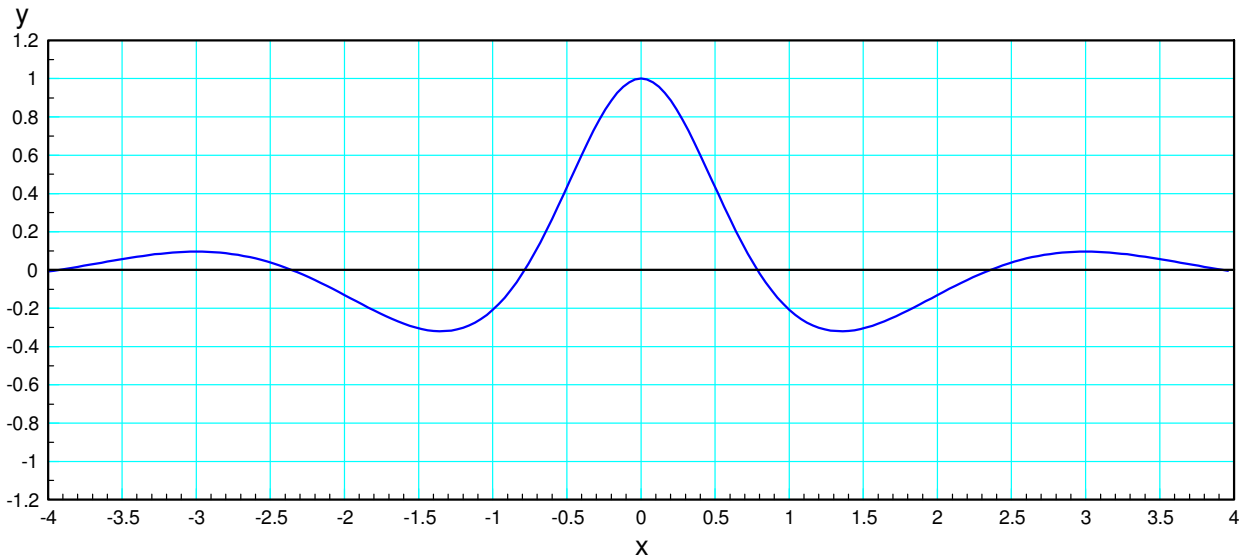Step 3: Plot the slopes and connect them with a smooth curve

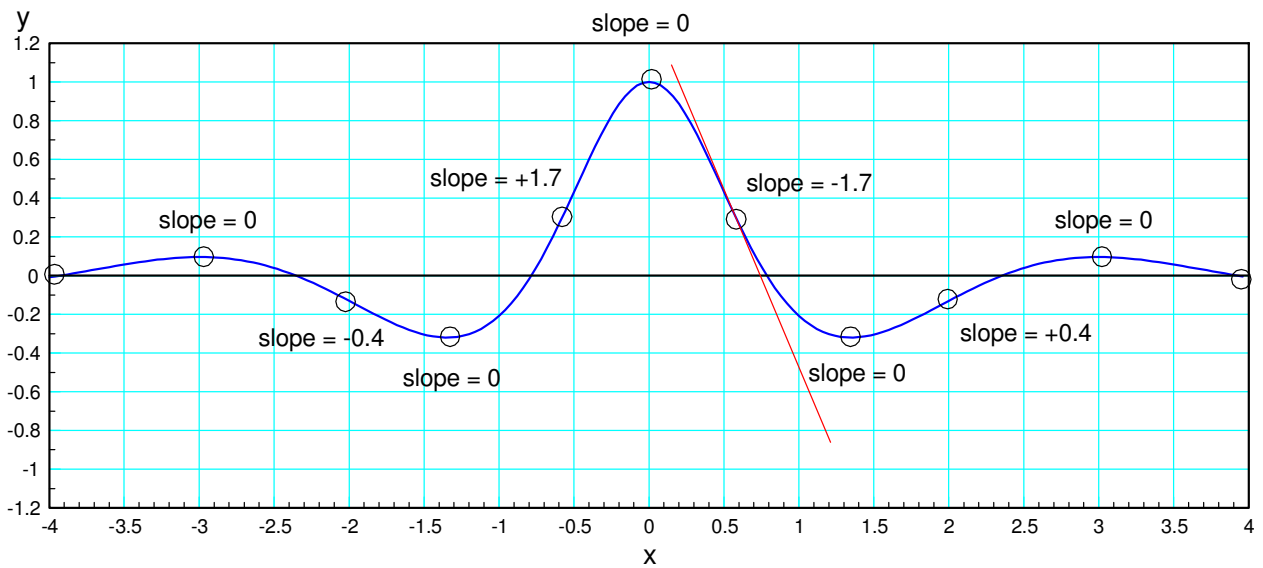The result (shown in red) is a cosine function

Graphical Example #3:

One nice thing about graphical methods is it works even if you can't find the derivatives using calculus. For example, find the derivative of
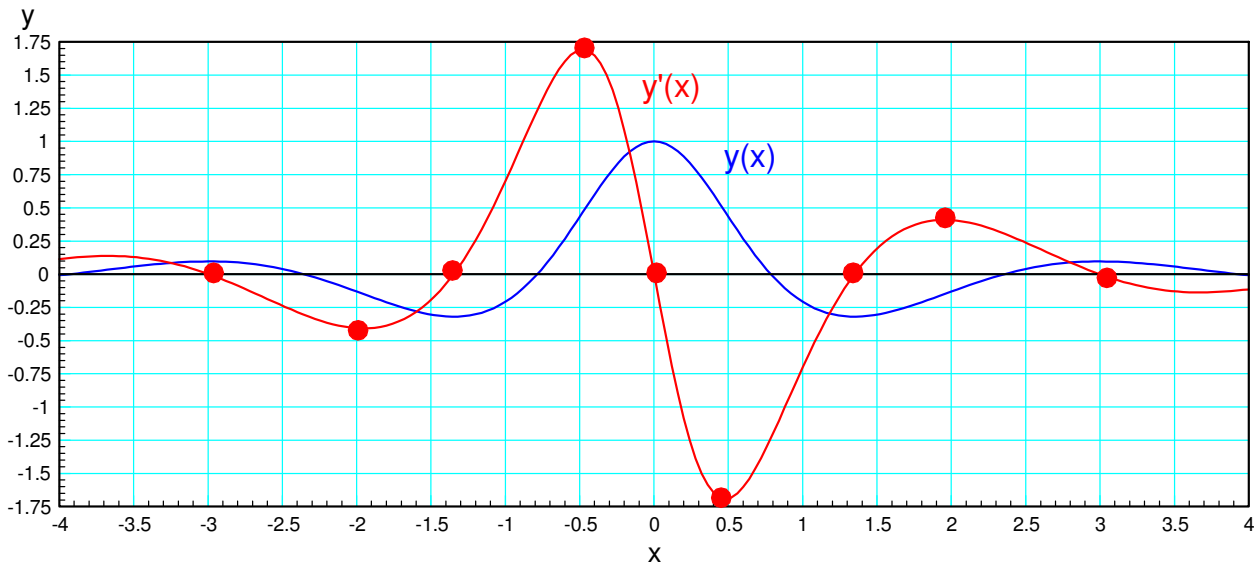
$$y = \frac{d}{dx}\left(\frac{\cos(x)}{x^2+1}\right)$$

Step 1: Compute the slope at several points

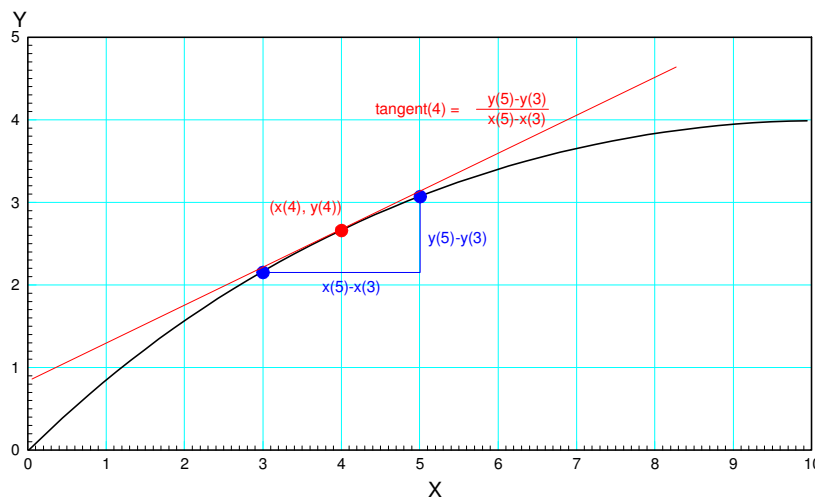Step 2: Plot the derivatives and connect with a smooth curve

Numerical Differentiation.

Repeating, the derivative at any given point is defined as

$$\frac{dy}{dx} \equiv \frac{\text{the change in y}}{\text{the change in x}}$$

If you have data for y(x), you can approximate the derivative at any point, say y(4), as

$$\left(\frac{dy}{dx}\right)_{x=4} \approx \frac{y(5)-y(3)}{x(5)-x(3)}$$

The derivative at x=4 can be computed as the change in y over the change in x

If you have a whole bunch of points, this can be done at a generic point in Matlab
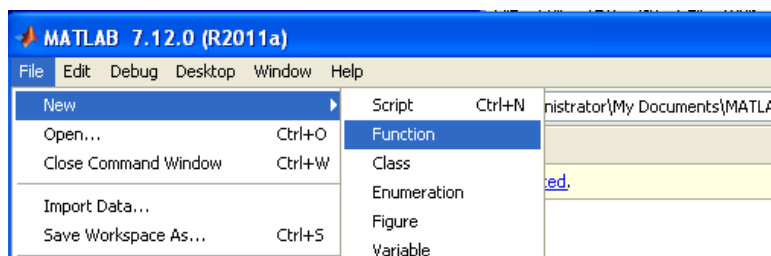
$$dy(i) \approx \frac{y(i+1)-y(i-1)}{x(i+1)-x(i-1)}$$

where 'dy' really means $\frac{dy}{dx}$ - but that's not a valid variable name in Matlab. If you put this in a loop, you can compute the derivative at any point.

Note that you'll have problems at the endpoints: the derivative computation uses the previous and next data point - which don't exist at the endpoints. For these, you'll need a slightly different computation.

One of the strengths of Matlab is you can create your own functions. To illustrate this, let's create a function, derivative' which is passed x and y and returns $\frac{dy}{dx}$.

In Matlab, click on 'File New Function'

Creating A New Function in Matlab: File New Function

In the editor window, type in the following:

```
function [dy ] = derivative( x, y )
% function [dy ] = derivative( x, y )

npt = length(x);

dy = 0*x;

for i=2:npt-1
    dy(i) = ( y(i+1) - y(i-1) ) / ( x(i+1) - x(i-1) );
end

dy(1) = (y(2) - y(1)) / ( x(2) - x(1) );
dy(npt) = (y(npt) - y(npt-1)) / ( x(npt) - x(npt-1));

end
```
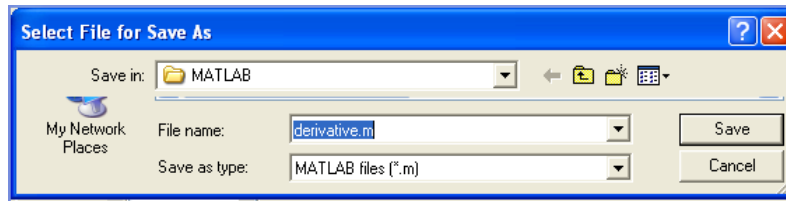
Save as 'deravitive.m'

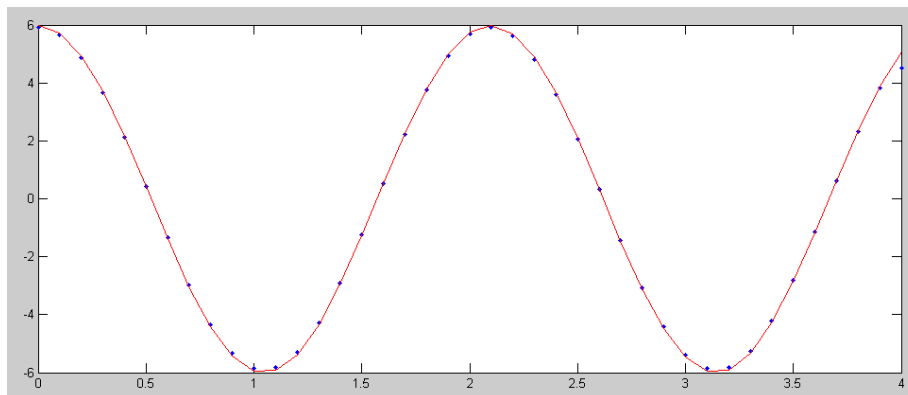Any time you create a new function, it's wise to test it on something where you know the answer. From Math 165,

$$\frac{d}{dx}(a\sin(bt)) = ab\cos(bt)$$

or

$$\frac{d}{dx}(2\sin(3t)) = 6\cos(3t)$$

Checking in Matlab:

```
>> x = [0:0.1:4]';
>> y = 2*sin(3*x);
>> dy = derivative(x,y);
>> plot(x,dy,'b.',x,6*cos(3*x),'r');
```


Comparison of the actual derivative of 2sin(3t) (red) and its computed value (blue dot).

From this plot, it appears that our function works - but has a slight error at the last data point. This is common - the first and last data points typically have problems when computing the derivative.

Note that a huge advantage of numerical differentiation is its ease of use: as long as you can get y(x) into Matlab, you can compute the derivative, y'(x). Even if you can't compute the derivative by hand.

For example, find the derivative of

$$y = e^{-x^2} \cdot \cos(x^3)$$

This looks intimidating, but it's not a problem with numerical differentiation.
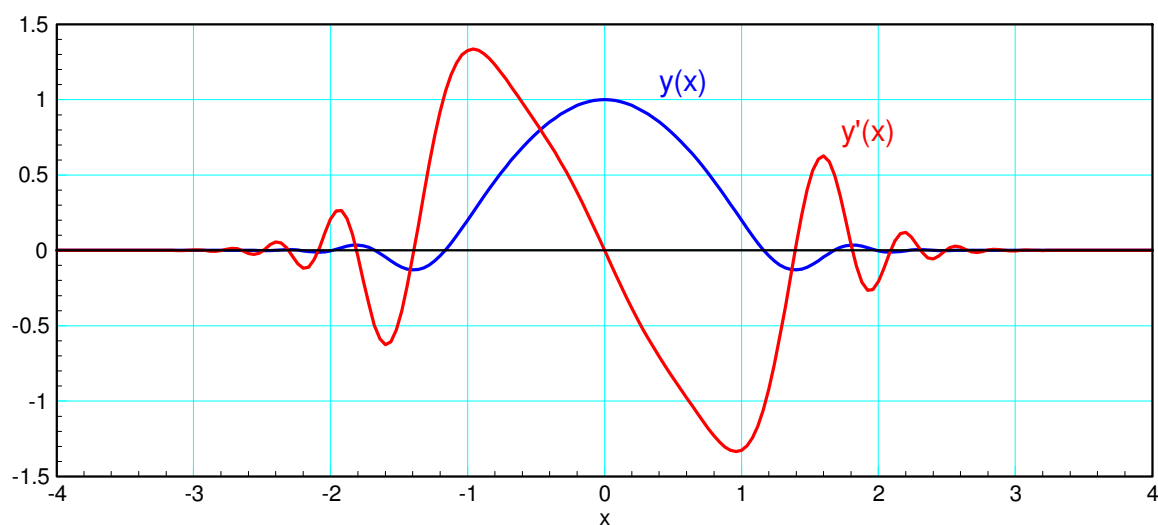
First, input y(x) into Matlab

```
>> x = [-4:0.04:4]';
>> y = exp(-x.^2) .* cos( x.^3);
>> plot(x,y)
```

Now find the derivative: dy/dx

```
>> dy = derivative(x,y);
>> plot(x,y,'b',x,dy,'r')
```

Matlab doesn't care what the funciton is: as long as you can get y(x) into Matlab, Matlab can find the derivative.


y(x) and its derivative using numerical differentiation

Note also that the derivative tells you when y is a maximum or a minimum: the maximum & minimum happen when y' = 0. This is one of the tricks you'll use in Calculus to determine the maximum and/or minimum of a function.

- Take the derivative of y(x)
- Determine when y'(x) = 0
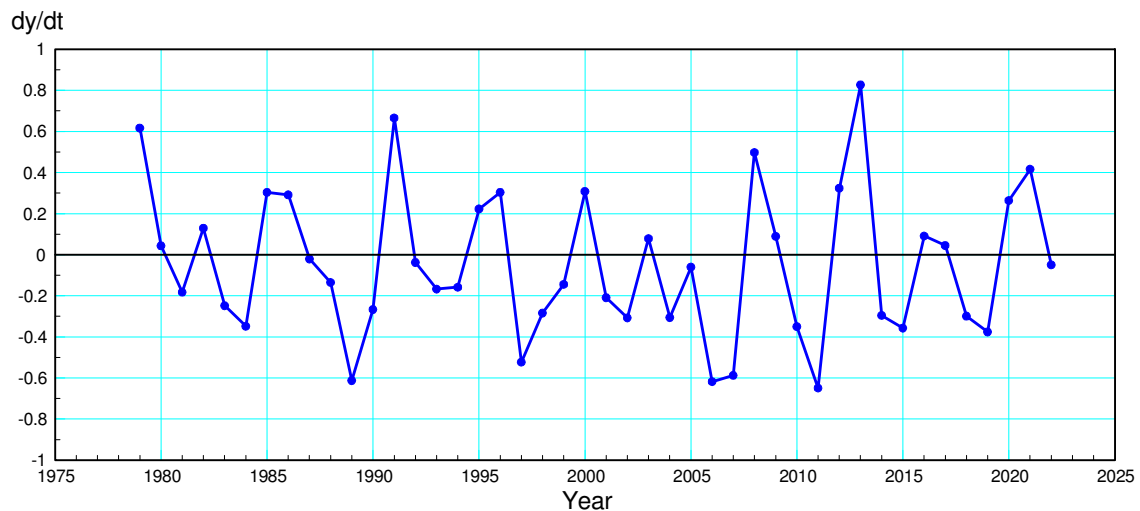- The maximum and minimum happen when y'(x) = 0.

Differentiation and Noise

One major problem with differentiation and real data is noise: random fluctuations that appear in the data. For example, if you look at Arctic ice levels, the data looks fairly clean:
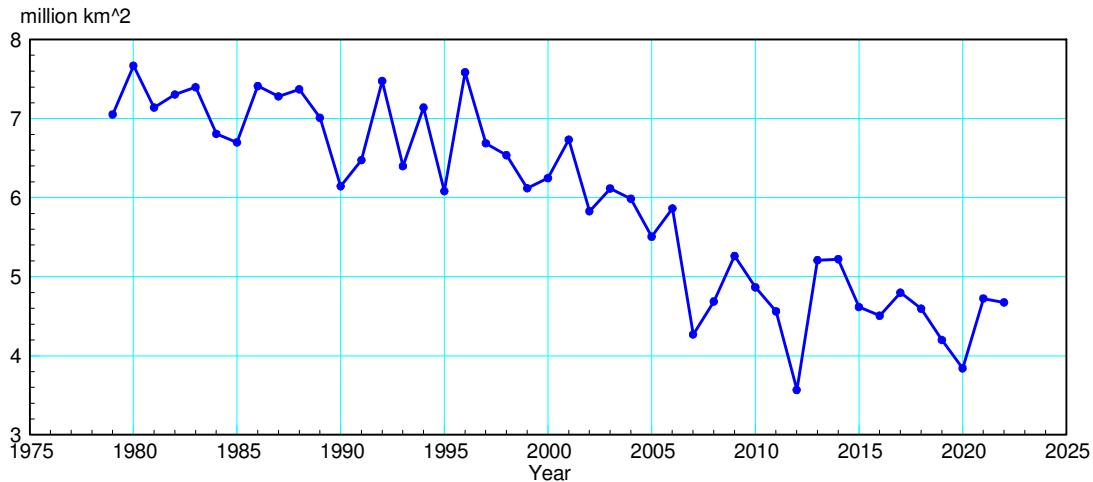

Arctic Sea Ice Area

If you plot the derivative, however, you have pretty much plain noise.

Change in Arctic Sea Ice Area

The derivative should contain interesting and useful information, such as what affect different global initiatives have on the Arctic. It's difficult to see this information through all the noise, however.

As a second example, consider global temperatures since 1970:
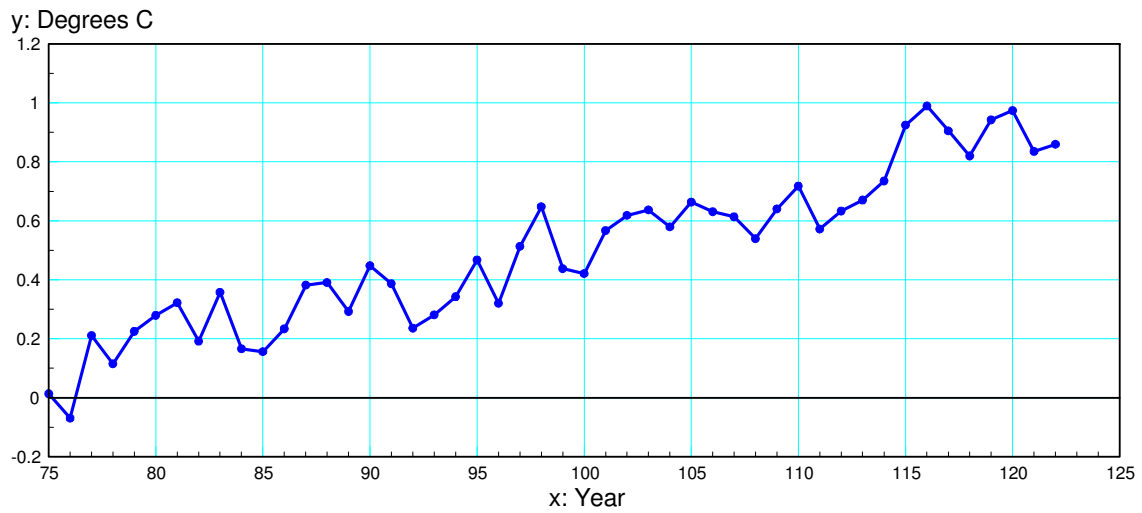
Global Temperature Deviation vs Year

The data looks fairly clean with a clear trend. The derivative would be interesting to see: it tells you how much the temperature is changing each year and what affects global initiatives have.

Computing the derivative and plotting it, however, pretty much just gives noise: there might be information here, but it's hard to see it through all the random fluctuations.
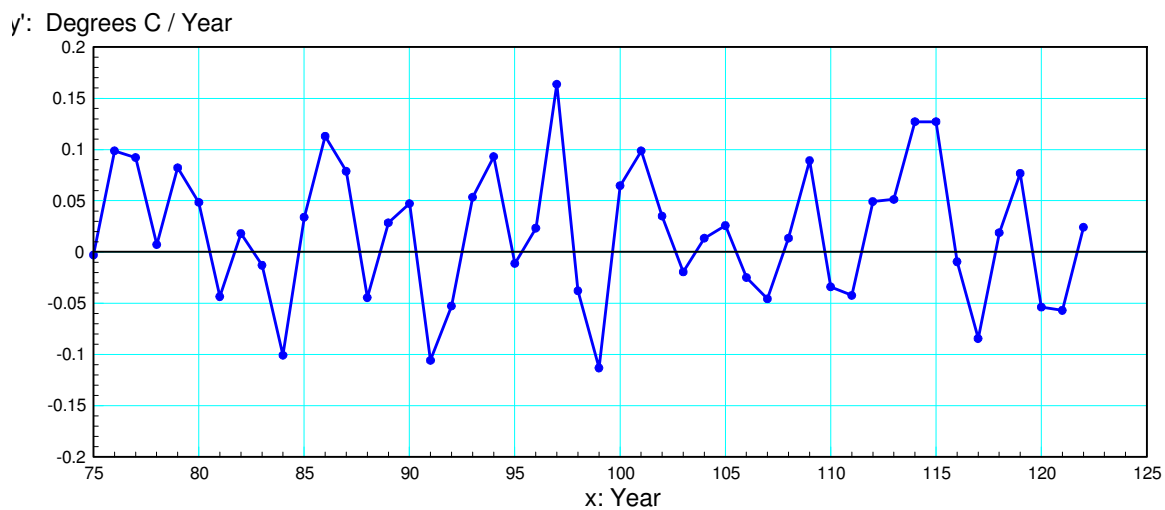
Change in Global Temperatures Each Year

Robotics & Path Planning

Yet another application of differentiation is in the area of robotics and path planning. For example, suppose you have a robotic arm that is to move from point A to point B. To do so, the motors must rotate, bending each joint.
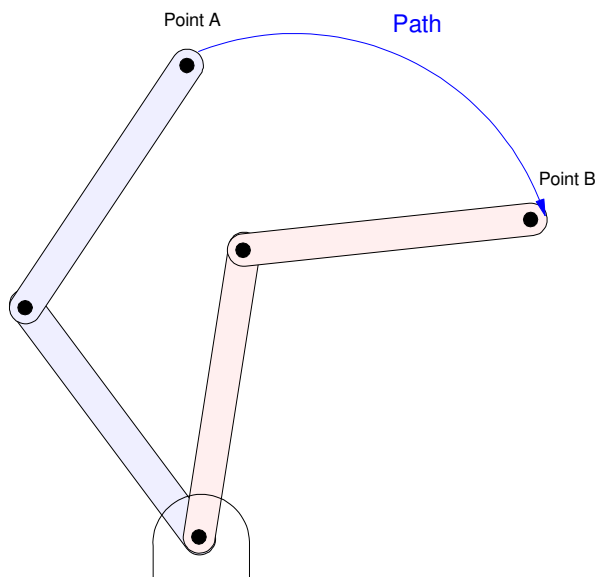
Robotic Arm: Drive the tip of the arm from point A to point B along some path

Just to simplify things, assume a given motor is to turn from an angle of 0 to 1 in two seconds. Path planning is how you do that.

Option 1: Linear Motion.

Assume motor follows the path

$$y = \begin{cases} 0 & t < 0 \\ t/2 & 0 < t < 2 \\ 1 & t > 2 \end{cases}$$

- The derivative of y tells you the motor's velocity. For DC motors, velocity is essentially voltage.
- The second derivative of y tells you the motor's acceleration or torque. For DC motors, torque tells you current.
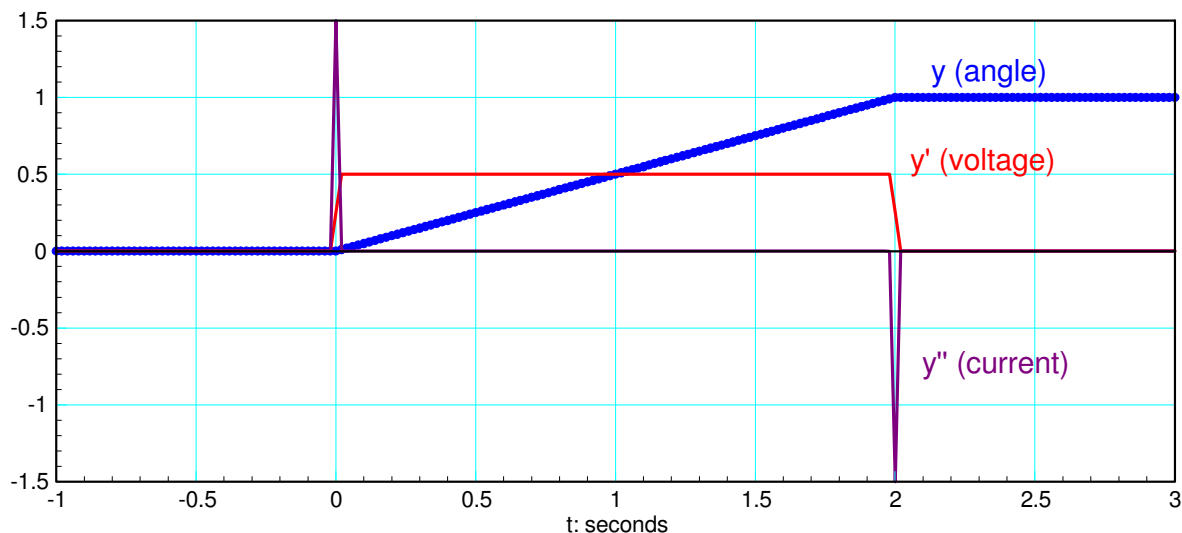
In Matlab, these can be computes as

```
>> t = [-1:0.01:3]' + 1e-6;

>> y = 0*(t<0) + (t/2) .* (t>0).*(t<2) + (1)*(t>2);
```

```
>> dy = derivative(t,y);
>> ddy = derivative(t,dy);

>> plot(t,y,t,dy,d,ddy)
```

While angle (y) looks OK, the acceleration (ddy) is a delta function, meaning the current required goes to +infinity and -infinity. Infinite current is hard on motors.
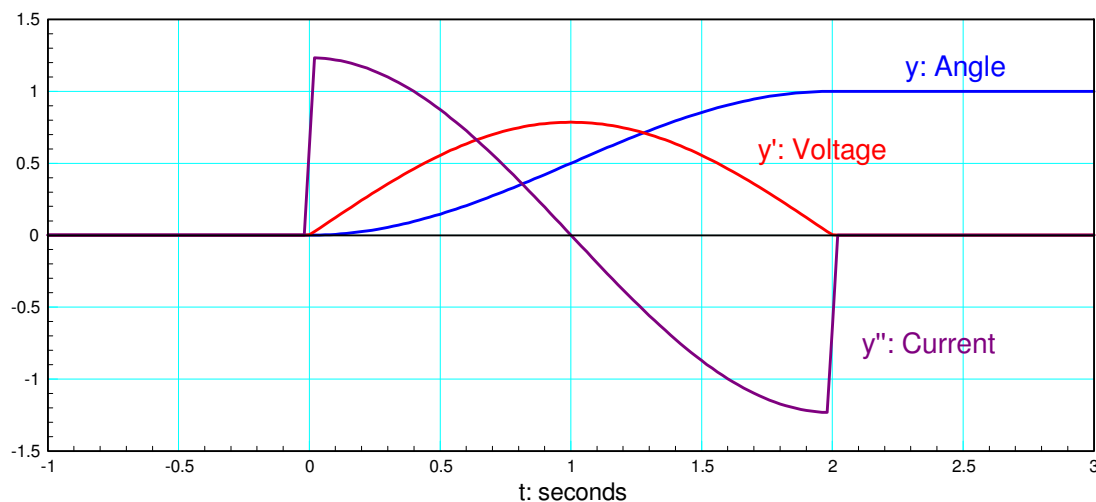

Option 2: Cosine Motion

Instead, let y go from 0 to 1 along the following path:

$$
y = \begin{cases}
0 & t < 0 \\
\left(\frac{1}{2}\right)\left(1 - \cos\left(\frac{\pi t}{2}\right)\right) & 0 < t < 2 \\
1 & t > 2
\end{cases}
$$

In Matlab:

```
>> t = [-1:0.01:3]' + 1e-6;
>> y = 0*(t<0) + ((1-cos(pi*t/2))/2) .* (t>0).*(t<2) + (1)*(t>2);
>> dy = derivative(t,y);
>> ddy = derivative(t,dy);
>> plot(t,y,t,dy,t,ddy)
```

This path is much more reasonable: the current remains finite (good).

Other paths from A to B can be defined. With differentiation, you can see what voltages and currents are implied with that path.

Summary:

Derivatives and differentiation are useful:

- If looking at the balance in a bank account, the derivative tells you when money is being added (derivative is positive) and withdrawn (derivative is negative). It also tells you how fast money is being added or removed.
- If looking at sea ice levels, the derivative tells you the amount of melt each year.
- If looking at DC motors, the derivative tells you the voltage (1st derivative) and current (2nd derivative).

Using graphical methods, you can sketch the derivative of a function

- The derivative at any point is the slope at that point

Using numerical methods (i.e. Matlab), differentiation is really easy. If you can get the function into Matlab, then Matlab can find the derivative.

- Finding an explicit expression for the derivative might be harder, however. That's something you'll cover in Calculus I.